Market Efficiency and Price Formation when Dealers are Asymmetrically Informed

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Abstract

We consider the effect of asymmetric information on price formation process in a quote-driven market where one market maker receives a private signal on the security’s fundamental. A model is presented where market makers repeatedly compete in prices: at each stage a bid-ask auction occurs and the winner trades the security against liquidity traders. We show that at equilibrium the market is not strong-form efficient until the last stage. We characterize a reputational equilibrium in which the informed market maker will affect market beliefs, possibly misleading them, in the sense that he will...

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push the uninformed participants to think the value of the risky asset is different from the realized one.

At this equilibrium a price leadership effect arises, quotes are never equal to the expected value of the asset given the public information, the informed market maker expected payoff is positive and the information revelation speed is slower than in an analogous order-driven market.

Keywords: bid-ask prices, asymmetric information, repeated auction, insider trading.

JEL Classification D82, D44, G10, G14.
1 Introduction

Several empirical studies show that different market makers either have access to different levels of information, or at least differ in their understanding of market fundamentals. In the foreign exchange markets, Peiers (1997) and de Jong et al. (1999) have shown that some commercial banks are indeed commonly considered to have some informational advantage due to their preferential relation with the central bank. In the bond market, Albanesi and Rindi (2000) detect some price leadership activity by large banks and consequently an imitative behavior of small banks. Indeed, large banks have a much larger customer base, so that their analysts will have a better view of the demand and the supply than small banks’ ones. These studies suggest two main implications: rst, that dealers often differ in their private perception of market fundamentals; second, that they know who are the best informed among them.

In the existing literature of financial microstructure, it is common to assume that private information is held by floor traders who submit anonymous orders to uninformed market makers. To the best of our knowledge, the case of asymmetric information among market makers has not been studied yet from a theoretical perspective.¹

There is an important difference between the asymmetric information among traders studied in the models à la Kyle (1985) and Glosten and Milgrom (1985), and asymmetric information among market makers. In the rst case informative orders cannot be separated from uninformative liquidity orders. Therefore, uninformed agents extract information observing the volume of trade that is just a noisy signal of the informed traders’ activity. By contrast, in a quote driven market, the quotes posted by market makers are perfectly observable by all market participants, and thus market makers can extract information from the quotes posted by the best informed among them.

We can expect that the strategies adopted by informed and uninformed agents in these two cases are substantially different. When orders are anonymous, as in Kyle (1985), an informed trader hides his activity behind noise traders, so that in equilibrium he can use simple monotonic strategies with-

¹Gould and Verecchia (1985) consider the case of a monopolist specialist that has private information on market fundamental. However, they consider a static game, and their result is obtained assuming that the specialist can precommit to add an exogenous noise to its price.
out revealing completely his information. By contrast, the transparency of quotes makes difficult for the informed dealer to exploit his advantage without revealing it, but it makes easier for him to influence uninformed agents. What is the net effect on the informational efficiency of the market? How can a market maker with exclusive private information optimally exploit his informational advantage in a quote driven market? What are the effects on quotes volatility and on the evolution of the bid-ask spread?

In this paper we study the price formation process and the efficiency of a quote driven market when: i) one of the market makers has superior information on the value of the traded asset; ii) his quotes are observable by the other market makers.

We consider a model where a risky asset is exchanged for a riskless asset between market makers and liquidity traders and where all market makers’ past quotes are observable. In each period, market makers simultaneously set quotes and automatically execute liquidity traders’ market orders. Notice that in some real markets the microstructure of exchanges is quite similar to our model. For example, figure 1 is a screen shot of what a Nasdaq dealer can see on his computer. In each instant dealers know who is proposing the best ask (or bid) price and what are the other proposed prices. Moreover in Nasdaq’s screen-based order routing and execution systems as SelectNet and the Small Order Execution System (SOES), orders of clients are automatically executed against market makers at the inside quotes. We quote from a document of NASD Department of Economic Research:

“Nasdaq market makers have also been subject to an increasing level of mostly affirmative obligations. Market makers must continuously post two-sided quotes, good for 1000 shares [...]; they must report trades promptly; they must be subject to automatic execution against their quotes via SOES; [...].” (J. W. Smith, J. P. Selway III, D. Timothy McCormick, 1998-01, page 2).

We assume that one of the market makers is informed about the liquidation value of the risky asset and, at some future date T + 1, this information will be publicly announced. The quantity exchanged in each period is constant and there is no exogenous shock coming from noise traders or from the arrival of new information. In each period, uninformed market makers extract information on the value of the asset observing the past quotes posted
Figure 1:

by the informed market maker. The latter takes into account the impact that his current quotes will have on the future uninformed dealers’ quoting strategy. The microstructure we model is substantially different from the models à la Kyle or Glosten and Milgrom. First, in our model private information is held by one of the dealers that is responsible to setting prices, whereas in the existing literature informed agents are traders who set quantities. Secondly, in our model the best informed agent’s action is perfectly observable and there is no exogenous shock coming from noise traders. By contrast, in the existing literature, informed traders’ orders melt with the exogenously random orders of noise traders.

We characterize market makers’ equilibrium quoting strategies in a one-period trading setting, and then we construct an equilibrium of the multiple periods case.

Our first result concerns the informational efficiency of the market. We show that in the last trading period market maker’s private information is fully revealed by his quotes but the probability that this revelation occurs earlier in time is less than one. In other words, the market is strong form efficient in the long run but not in the short run.\(^2\) Combined with the result

\(^2\)This result extends to any distribution of the liquidation value of the asset and to the
of Flood et al. (1998), where they show that efficiency is greatest in the most transparent trading mechanism, we argue that our result should extend to Nasdaq if dealers are given the option to submit anonymous quotes, and to anonymous markets as “Telematico” for fixed income securities.

Moreover, we show that in equilibrium the informed market maker generates endogenously some “noise” in his quoting activity, that precludes the others to infer immediately his private information.

The intuition of this result relies on two observations: i) if the value of the asset is high it is worth buying it by setting high bid quotes, whereas if the value of the asset is low it is worth selling it by setting low ask quotes; ii) the more correct is the uninformed dealers’ belief, the smaller will be the profit for the informed market maker as the trading prices will be closer to the true value of the asset. Thus, on one hand, when the informed market maker chooses the quotes that maximizes his current profit, he reveals part of his information and decreases his future profit. On the other hand, if he chooses quotes that make him lose money in the current trade, he will increases his future profit by misleading the uninformed market makers. In the last trading period, this trade-off vanishes, the informed dealer simply takes the action that maximizes his current profit, and so his quotes fully reveal the value of the asset. However, in the periods before the last, it is optimal for the informed market makers to randomize between revealing his information and misleading. In this way he can exploit his information advantage for several trading periods despite his quotes are perfectly observable.

We also provide some empirical implications of this equilibrium.

First, quotes are volatile despite there is no exogenous shock during the trading process. Indeed market makers’ quotes move because the uninformed dealers’ belief change and because in every period they are the outcome of a mixed strategies.

Second, in equilibrium the inside spread is always non-negative and the average market spread increases as the game reaches its end. This last result explains the empirical observation that spread increases when the date of the introduction of price dependent trader’s demand (see Calcagno and Lovo, (1998)).

3The more uninformed dealers’ belief is correct the smaller will be the difference between the true value of the asset and its expected value.

4This strategy brings to mind the reputation effect pointed out by Kreps and Wilson (1982). When a player has a doubt about his opponent’s type, the latter can manage to build a misleading reputation by copying the strategy which would be optimal for a type different from his own.
public report approaches. This is in contrast with both Glosten and Milgrom (1985) and Kyle (1985) where in equilibrium the depth of the market is respectively decreasing or constant across time.

Third, we find that the equilibrium presents a positive serial correlation between the quotes set by the informed dealer at time $t$ and the quotes set by the uninformed market maker at time $t+1$. This is in tune with the empirical evidence obtained in Peiers (1997) for the foreign exchange market, where large German banks appear to be price leaders while there is a group of banks that lag behind the market.

Fourth, we can measure the speed of information revelation, and compare it with the Kyle model. We find that the quote-driven structure we have modelled performs worse in terms of informational efficiency than the order-driven structure of Kyle (1985). Indeed, the conditional variance of the asset given the observable information decreases with a lower rate than in the Kyle model.

The remainder of this paper is organized as follows. Section 2 presents the formal model. In section 3 we collect the construction of the equilibrium in the one, two, and $T$ steps case, and we prove the short run information inefficiency of the equilibrium in the strong-sense. In section 4 we derive some empirical predictions from the properties of a numerical solution of the model. In section 5 we concludes, and all proofs are collected in the Appendix.

2 The model

Consider a market with $N$ risk-neutral market-makers (MMs in the following) who trade a single security over $T$ periods against liquidity floor traders. Each period market makers set bid and ask prices, which are rm for a given quantity of the security$^5$.

The liquidation value of the security is a random variable $\mathcal{V}$ which can, for simplicity, take two values, $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_1 > \mathcal{V}_2$; according to a probability distribution $(p; 1-p)$ commonly known by all MMs, where $p = \Pr(\mathcal{V} = \mathcal{V}_1)$. We denote $v = p\mathcal{V}_1 + (1-p)\mathcal{V}_2$ the expected value of the asset for any given $p$. The realization of $\mathcal{V}$ occurs at time 0 and at time $T+1$ a public report will announce it to all market participants. Time is discrete and $T$ is finite.

$^5$This is the case, for example, in some Nasdaq’s execution systems (see the introduction).
Information structure

At the beginning of the first period of trade, one of the MMs, MM1, is privately informed about the realized liquidation value of the risky asset. Following an usual convention in games with incomplete information, we will refer to the realization \((V; \bar{V})\) as the “type” of MM1, and call MM1(\(V\)) (resp. MM1(\(\bar{V}\)) the informed MM when \(\Psi = V\) (resp. \(\Psi = \bar{V}\)). The other \(N - 1\) market makers do not observe any private signal but they know that MM1 has received a superior information; we will treat them as a unique dealer called MM2. In each period every market maker can observe the past quotes of all market makers.

Market Rules

In each period the two MMs simultaneously announce their asks and bid quotes which are \(\cdot \)rm for one unit of the asset. Then, transactions take place between liquidity traders and the market makers. We assume that at each date, liquidity traders sell one unit of the asset to the market maker who set the highest bid, and buy one unit of the asset from the market maker who sets the lowest ask (i.e. price priority is enforced). If both market makers set the same quote, liquidity traders are indifferent in their trading counterpart and we assume that they will exchange with MM2. Finally, we assume that market makers can not trade with each other and that short sales are permitted.

Behavior of market participants and equilibrium concept

In each period a buy market order and a sell market order are proposed by floor traders who trade for liquidity reasons. It is worth stressing that in our

\(^6\) As in Kyle (1985) we assume that there is only one agent that receive private information on the realization of \(V\).

\(^7\) This assumption is made without loss of generality because the informed market maker only considers the probability of winning the auctions at a given price, no matter if this probability is the outcome of the strategy of one uninformed player or equally uninformed players (see also Engelbrecht-Wiggans et al. (1982) and section 3.1).

\(^8\) For simplicity we do not consider the timing problem arising when the bidding process is sequential, as in Cordella and Foucault (1998).

\(^9\) It is standard in the literature to \(\cdot \)x the traded quantity in each step (see O’Hara (1995)), and as we said before this assumption captures quite closely the rules of some markets.

\(^10\) As market makers are risk neutral, this is equivalent to assume that in each period there is a constant probability of observing a buy order or a sell order.

\(^11\) This assumption simplifies the notation.
model traders do not act for informational motives, and so the flow of market orders neither incorporates, nor depends on any information about the value of the asset. As price priority is enforced in any period, each market maker knows that he will buy (resp. sell) one asset only if he proposes the best bid (resp. ask) quote. We denote $a_{i;t}$ and $b_{i;t}$ the ask and bid price respectively set by market maker $i$ in period $t$. Assuming that market makers are risk neutral, we can write the single period payoff functions for market makers as follows:

$$
\Pi_{1;t}(V) = (a_{1;t} - V) \Pr(a_{2;t} > a_{1;t}) + (V - b_{1;t}) \Pr(b_{2;t} < b_{1;t}) \\
\Pi_{1;t}(V) = (a_{1;t} - V) \Pr(a_{2;t} > a_{1;t}) + (V - b_{1;t}) \Pr(b_{2;t} < b_{1;t})
$$

for $MM_1(V)$ and $MM_1(V)$ respectively, and for $MM_2$

$$
\Pi_{2;t} = p(a_{1;t} - V) \Pr(a_{2;t} \in \Phi) + (1 - p)(a_{1;t} - V) \Pr(a_{2;t} \in \Phi) + p(V - b_{1;t}) \Pr(b_{2;t} \in \Phi) + (1 - p)(V - b_{1;t}) \Pr(b_{2;t} \in \Phi)
$$

for $MM_2(V)$ and $MM_2(V)$ respectively, and for $MM_2$

The overall payoff of each MM is simply the (non discounted) sum for $t = 1, \ldots, T$ of these payoffs:

$$
\Pi_{1;t}(V ; T ; p) = \sum_{t=1}^{T} \Pi_{1;t}(V) \quad \text{for} \quad V = fV ; Vg:
$$

$$
\Pi_{2;t}(V ; T ; p) = \mathbb{E}[\Pi_{2;t}]
$$

The quotes that MMs post at date $t$ could in principle depend on the past quotes. For tractability, we restrict to equilibria where the MMs’ strategy are Markov strategies, which depend only on the state of the game $\sigma_t = (T - 1 + t; p_t)$, that is defined by the number of trading rounds before the public report ($T - 1 + t$) and the uninformed dealer’s belief $p_t$. Given this restriction, a mixed strategy for $MM_2$ in period $t$ can be defined with a function $\Pi_2$ that maps the state of the game $\sigma_t$ into a probability distribution over all couples of bid-ask quotes. As $MM_1$’s strategy depends also on his

\footnote{MMs could use more complex strategies which depend on the whole set of past quotes, or at least on a bigger subset of them than in the Markov case. These strategies are extremely complex to analyze in our framework, and this puts a serious restriction to their actual implementability.}
private information, a mixed strategy for \( \text{MM}_1 \) in period \( t \) is a function \( \frac{\gamma}{4} \) that maps the value of the asset and the state of the game \( \omega_t \) into a probability distribution over all couples of bid-ask quotes. For a given state of the game \( \omega = (\zeta; p) \) we denote \( \frac{\gamma}{4}(V; \zeta; p) \) and \( \frac{\gamma}{4}(\zeta; p) \) the expected equilibrium payoff for \( \text{MM}_1 \), given \( \Psi = V \), and for \( \text{MM}_2 \) respectively.

We characterize the equilibrium strategies \( \frac{\gamma}{4} \) and \( \frac{\gamma}{4} \) solving the game by backward induction: at any time \( t \) MMs solve the following problems:

\[
\begin{align*}
\frac{\gamma}{4}(V; \zeta; p_t) &= \arg\max_{\frac{\gamma}{4}(V)} \frac{\gamma}{4}(V; \zeta; 1; p_{t+1}), \text{ given } \frac{\gamma}{4} \\
\frac{\gamma}{4}(\zeta; p_t) &= \arg\max_{\frac{\gamma}{4}(\zeta)} \frac{\gamma}{4}(V; \zeta; 1; p_{t+1}), \text{ given } \frac{\gamma}{4} \\
\frac{\gamma}{4}(\zeta; p_t) &= \arg\max_{\frac{\gamma}{4}(\zeta)} \frac{\gamma}{4}(\zeta; 1; p_{t+1}), \text{ given } \frac{\gamma}{4}
\end{align*}
\]

where \( \zeta = T + 1 - t \) and \( p_{t+1} = \Pr(\Psi = V_j a_{t+1} b_{t+1}) \) is determined by the Bayes rule when this is possible and it is arbitrarily chosen otherwise.

We denote \( i(T; p) \) the game representing the strategic interaction among MMs when there are \( T \) finite rounds of trade and \( \Pr(\Psi = V) = p \) at the beginning of the game \( t = 0 \).

To sum up, in each period market makers compete simultaneously in two first price auctions. They compete to buy one unit of the asset from a liquidity trader (in the bid auction) and to sell one unit of the asset to another liquidity trader (in the ask auction). Intuitively when \( \Psi = V \) (resp. \( \Psi = V_j \)), it is worth buying (resp. selling) the asset rather than selling (resp. buying) it, and so, the bid (resp. ask) auction is profitable and the ask (resp. bid) auction is not.

Observing the quotes posted by \( \text{MM}_1 \) in the past trading periods, \( \text{MM}_2 \) tries to understand which side of the market is profitable. Thus \( \text{MM}_1 \) faces a trade-off between trying to win the profitable auction and revealing his information.

Notice that if at some \( t \), \( \text{MM}_2 \) learns the true value of the asset, then the asymmetric information vanishes, the market makers compete à la Bertrand, bid and ask quotes coincide with the true value of the asset and all market makers’ payoffs are zero.

It is worth stressing that as market makers can alternatively buy or sell the security without inventory considerations, there is always one of the two auctions that is profitable and one that is not, no matter the true value of the
asset. This suggests a symmetry property of the game. In the appendix we formally state this symmetry between the bid and ask auction, that we now explain intuitively. First it is always possible to rename market participants and strategies such that one can obtain a game that describes an ask auction starting from the game describing a bid auction and vice versa. Second, what really matters for the equilibrium of the game is not the actual value of the asset, \( V \) or \( V' \), but how close is the belief of M M 2 to the truth: intuitively the more correct are M M 2’s belief, the smaller is M M 1’s pro..t.

3 Equilibrium characterization

3.1 One trading round

In this section we analyze the dealers’ price competition when \( T = 1 \), which can also be interpreted as the last trading round. The bid auction alone has been studied by Engelbrecht-Wiggans, Milgrom and Weber (1983) (EMW henceforth) for an arbitrary distribution of the asset for sale. They show that the equilibrium is unique and fully revealing, in the sense that M M 2 can infer unambiguously the value of the asset after observing M M 1’s quotes.

Proposition 1 extends their result to the ask auction. Moreover it provides the equilibrium distribution of bid and ask quotes and market makers’s equilibrium payoff for our specification of the traded asset’s distribution.

Proposition 1 The equilibrium of the one shot game \( \langle 1; p \rangle \) is unique and it is such that:

(i) M M 2 randomizes ask and bid prices according to

\[
\Pr(a_{2,1} < x) = F^2(x) = \begin{cases} 
0 & \text{for } x \in [V'; v] \\
\frac{x \wedge v}{x \wedge V} & \text{for } x \in [V; V'] \\
1 & \text{for } x \in [v; 1]
\end{cases}
\]

\[
\Pr(b_{2,1} < x) = G^2(x) = \begin{cases} 
0 & \text{for } x \in [V'; v] \\
\frac{v \wedge x}{v \wedge V} & \text{for } x \in [V; v] \\
1 & \text{for } x \in [v; 1]
\end{cases}
\]
(ii) If the value of the asset is $\bar{V}$, then $\text{MM}1$ sets $b_{1;1} = \bar{V}$ and he randomizes the bid price according to

$$\Pr(b_{1;1} : x \mid \Psi = \bar{V}) = \mathcal{G}^{\bar{V}}(x) = \begin{cases} 0 & \text{for } x \in [1; 1 ; \bar{V}] \\ \frac{1}{p(\bar{V} \mid x)} & \text{for } x \in [\bar{V} ; v] \\ 1 & \text{for } x \in [v ; 1] \end{cases}$$

(iii) If the value of the asset is $v$, then $\text{MM}1$ sets $b_{1;1} = v$ and he randomizes the ask according to

$$\Pr(a_{1;1} : x \mid \Psi = v) = \mathcal{F}^{v}(x) = \begin{cases} 0 & \text{for } x \in [1; v] \\ \frac{x - v}{1 - p(\bar{V} \mid x)} & \text{for } x \in [v ; V] \\ 1 & \text{for } x \in [V ; 1] \end{cases}$$

(iv) Equilibrium payoffs are $\frac{1}{2}(1; p) = 0$, $\frac{1}{2}(\bar{V}; 1; p) = (1 - p)(\bar{V} ; v)$ and $\frac{1}{2}(v; 1; p) = p(\bar{V} ; v)$.

One period before the public report, the informed market maker has a last opportunity to gain from his private information and he does not care about $\text{MM}2$’s posterior beliefs. More concretely, if the liquidation value of the asset is $\bar{V}$, $\text{MM}1$ will try to buy the asset by winning the bid auction, whereas if the liquidation value of the asset is $v$, he will try to sell the asset by winning the ask auction. The uninformed market maker does not know whether it is profitable to buy or to sell the asset, and so he will try to win both auctions.

The discrete distribution of $\Psi$ implies that the equilibrium is in mixed strategy. This means that when a MM tries to buy (resp. sell) the asset he chooses his bid (resp. ask) price quotes using a lottery. In equilibrium bid quotes are distributed between $\bar{V}$ and $v$, whereas ask quotes are distributed between $v$ and $\bar{V}$.

To understand why a pure strategy equilibrium does not exist notice first that, $\text{MM}2$ can always guarantee a zero profit by setting $a_2 = \bar{V}$ and $b_2 = v$. For this reason, he never posts bid greater than $v$ or ask lower than $v$, as this would provide him with a strictly negative profit. This has two implications: first, $\text{MM}1$’s equilibrium payoff is strictly positive as he

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13 Indeed as $\text{MM}s$ set simultaneously their quotes, $\text{MM}2$ will deduce the actual value of the asset from $\text{MM}1$’s quotes only after having posted his own quotes, that is too late.
can always guarantee it; second, it is never optimal for \( MM_1 \) to post bid (resp. ask) strictly greater (resp. lower) than \( v \), when \( \Theta = V \) (resp. \( \Theta = V \)). Thus if a pure strategy equilibrium exists, then \( MM_1 \) would post \( a_1^*, v \) (with probability one) when \( \Theta = V \) and \( b_1^*, v \) when \( \Theta = V \). But in this case \( MM_2 \), best reply would be to post \( a_2^* = a_1^* \) and \( b_2^* = b_1^* \), and \( MM_1 \)'s equilibrium payo\( \text{f} \) would be zero, that contradicts the observation that his payo\( \text{f} \) is positive.

To sum up, in the static game, the asymmetry of information between market makers leads to three important implications. First, the full revelation of information by \( MM_1 \) makes the market strong-form e\( \text{f} \)cient at the last stage of trade. This follows from the fact that \( MM_1 \)'s quotes are observable.

Second, unlike the symmetric information case, bid and ask market prices are different from the expected liquidation value of the asset given the information available to market makers. Indeed market spread is typically positive and bid and ask quotes straddle \( v \). However, there is no restriction over its width (up to \( V \) which depends on the output of the mixed strategies.

Third, although the uninformed market maker expected equilibrium payo\( \text{f} \) is zero, the best informed market maker obtains a positive expected payo\( \text{f} \). More precisely, his informational rent is larger when the \( MM_2 \)'s belief is wrong (i.e. \( j e V - v j \) is large). Indeed, in this case, \( MM_1 \) can win the profitable auction at prices that are far from the true value of the asset.

### 3.2 Informational e\( \text{f} \)ciency of the quote-driven market

In the last trading period \( MM_1 \) reveals to the market his private information through his posted quotes. However, this is not true for any period before the last one. More precisely, we now show that the probability that private information is completely conveyed into prices before the last period auction is less than one.

Consider an equilibrium of \( (T; p) \) and let \( S_t(\Theta) \) be the support of the probability distribution of the private information, where

\[ S_t(\Theta) = \{ \Theta' \in \mathbb{R}^2 : \text{Pr}(\Theta' \in S_t(\Theta)) = 1 \} \]

for \( \Theta' \) in the support of the private information distribution.

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Footnotes:

14 For example, by setting \( a_1 = v + \epsilon \) if \( \Theta = V \) and \( b_1 = v - \epsilon \) if \( \Theta = V \), with \( \epsilon > 0 \), his profit can be arbitrarily close to \( j e V - v j > 0 \).

15 \( S_t(\Theta) \) is the smallest subset of \( \mathbb{R}^2 \) such that in equilibrium \( Pr((a_1,t,b_1,t) \in S_t(\Theta)) = 1 \), for \( \Theta' \) in the support of the private information distribution.
of bid and ask prices played in some period \( t \) by \( \text{MM}_1(\mathcal{V}), \mathcal{V} \leq f\mathcal{V};\mathcal{V}\).  

We say that a fully revealing phase occurs in period \( t \), if  
\[ S_t(\mathcal{V}) \setminus S_t(\mathcal{V}) = ;. \]

In this case \( \text{MM}_2 \) unambiguously understands the value of the asset by observing whether \((a_t; b_t)\) belongs to \( S_t(\mathcal{V}) \) or to \( S_t(\mathcal{V}) \). After a fully revealing phase, the true value of the asset is commonly known and the \( \text{MMs'} \) continuation payoffs are zero.

The following theorem shows that no fully revealing phase can occur before the end of the game. Therefore, the private information is never revealed with probability one before \( T \) and thus in the short run the market is not efficient in strong-form sense. In other words, in the short run, it is not possible to infer \( \text{MM}_1 \)'s private information despite his quotes are perfectly observable. This inefficiency result has the flavor of the results obtained in the existing microstructure models where there is imperfect information. However, contrary to what happens in these models (for example Kyle (1985)), our result does not rely on the noise exogenously generated by liquidity traders. Theorem 2 shows that when an informed dealer cannot hide behind noise traders or anonymity of actions, he will generate endogenously some noise in order to best exploit his private information.

**Theorem 2** In any Bayesian-Nash equilibrium, \( 8t < T \) the probability that \( \text{MM}_1 \)'s quotes fully reveals his private information in \( t \) is less than 1.

The intuition behind the proof is that a fully revealing phase is not credible before the last trading round. More precisely, if at equilibrium \( \text{MM}_1 \)'s private information was surely fully revealed at some \( t < T \), then market makers would play the unique equilibrium of the one shot game in \( t \). However, in this case, \( \text{MM}_1 \) has at least one profitable deviation. For example, if the value of the asset is \( \mathcal{V} \), \( \text{MM}_1 \) could post quotes that only \( \text{MM}_1(\mathcal{V}) \) would post according to the equilibrium in proposition 1, and then profit of this misleading activity in the following \( T - t \) trading periods.

We will now construct an equilibrium of \( (T; p) \) where \( \text{MM}_1 \) induces a gradual revelation of his information to the market and then derive some of its properties that can be empirically tested.

\[16\] In the last repetition of the game \( S_T(\mathcal{V}) = f(a_t; b_t) : a_t = \mathcal{V}, b_t \geq f[v; \mathcal{V}]; \mathcal{V} \), whereas \( S_T(\mathcal{V}) = f(a_t; b_t) : a_t \leq f[v; \mathcal{V}]; \mathcal{V} = \mathcal{V} \), and so it results \( S_T(\mathcal{V}) \setminus S_T(\mathcal{V}) = ;. \)
3.3 Manipulating strategies in equilibrium

The only way MM\textsubscript{1} can exploit his informational advantage for more than one period is by playing mixed strategies which originate some noise in his quotes. Concretely, this is possible randomizing across quotes that he would have posted also if the value of the asset was different.

From the analysis of the one period case, we already know that in the last trading stage, the informed MM competes only on the profitable side of the market: he tries to sell the asset if $\Psi = \Upsilon$ and to buy it if $\Psi = \Upsilon$. During the trading periods before the last one, in the equilibrium we characterize, MM\textsubscript{1} “hides” his information by participating to the unprofitable side of the market with positive probability. In this way, MM\textsubscript{2} cannot unambiguously deduce MM\textsubscript{1}’s information by observing whether MM\textsubscript{1} was trying to buy or to sell the asset in the previous period. We call this kind of strategies manipulating strategies as there is a positive probability that the informed MM takes an action that aims to turn the uninformed MM’s belief in the wrong direction.

MM\textsubscript{1}’s incentive to mislead MM\textsubscript{2} by trying to win the unprofitable auction depends on two factors: the benefit that a misleading action has on the future payoff on one hand, and the current cost of misleading on the other hand. Intuitively, the larger is the number of remaining trading periods, the larger will be the weight of the future payoff, and so the larger will be MM\textsubscript{1}’s benefit to mislead MM\textsubscript{2} in the current period. The cost of misleading depends on the correctness of MM\textsubscript{2}’s belief. For example, if $V = \Upsilon$ and MM\textsubscript{1} wants to mislead MM\textsubscript{2}, he will post ask prices close to $v$ so that he will sell the asset with positive probability\textsuperscript{17}. Then the cost of misleading can be measured by the difference between $\Upsilon$ and $v$. More precisely, we can measure the correctness of MM\textsubscript{2}’s belief with the variable $e = 1 - |\Psi - V|$, that is equal to 1 when MM\textsubscript{2} knows the true value of the asset and it is close to 0 when his belief is completely wrong\textsuperscript{18}. In general, the cost of misleading decreases with the correctness of MM\textsubscript{2}’s belief.

We can now describe in words the equilibrium strategies at $t = 0$ for any $T$. First, whenever a market maker tries to buy (resp. sell) the asset, he randomizes his bid (resp. ask) on the interval $[b_{\min}; v]$ (resp. $[v; a_{\max}]$), where

\textsuperscript{17}Intuitively, MM\textsubscript{2} will never accept to sell the asset at a price $a_2 < v$ so that MM\textsubscript{1} can be sure to win the ask auction with an $a_1$ sufficiently close to $v$.

\textsuperscript{18}Notice that if $\Psi = \Upsilon$ the $e = p$, whereas $e = 1 - p$ when $\Psi = \Psi$. 

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b_{min} and a_{max} depend on the state of the game (T; p). Second, the uninformed M M participates to both sides of the market trying to buy and sell the asset. Third, the strategy of M M 1 depends on the correctness of M M 2's belief. If the M M 2's belief is sufficiently wrong, namely when $e < 2^{1 - T}$, then M M 1 does not mislead and competes only on the pro...table side of the market. If $e > 2^{1 - T}$, then M M 1 randomizes between trying to win the pro...table side and trying to win the unpro...table one. In no case M M 1 buys and sells simultaneously the asset.

Two remarks follow from this description. First, the threshold $2^{1 - T}$ converges exponentially to 0 when T increases. This reflects the intuition that misleading occurs with positive probability for any given level of belief, provided that there are enough trading rounds before the public report. For this reason one should expect that informativeness of M M 1's quotes is low at time zero and increases when T approaches.

Second, if T is the number of trading periods before the public report, a M M 1's quotes fully reveal his information with positive probability if and only if $e > 1 - 2^{1 - T}$. This implies that full revelation might occur only when M M 2's belief is sufficiently correct.\footnote{See section 6.1 for an explanation of this result.}

In the following section we present the construction of the equilibrium for the game repeated twice. In the appendix we provide a formal statement for any length T of the game.

Given the game $\mathcal{G}((T; p)$ we denote as follows the equilibrium distribution of market makers' quotes during the first trading period: $G(b) = \Pr(b_{1:1} \cdot bj = v), G(a) = \Pr(a_{1:1} < aj = v), F(a) = \Pr(a_{1:1} < aj = v)$). Furthermore we denote $g(b) = G'(b), f(a) = F'(a)$.

### 3.3.1 The two-periods game

According to the qualitative description of the equilibrium given above, we can describe as follows MMs' quoting strategies in the first round of trade for the game $\mathcal{G}(T; p)$, when $T = 2$ and $p > 1/2$.

**Feature A)** In equilibrium, M M 2 tries both to buy and to sell the asset simultaneously by randomizing his bid and ask quotes on the support $[b_{min}; v] \cdot [v; a_{max}]$. If the value of the asset is $v$, then $e = 1 - p < 2^{1 - T} = 1/2$ and the informed market maker competes only in the pro...table auction. That is, he posts a bid price equal to $b_{min}$ and he randomizes the ask price.
in the interval \([v; a_{\max}]\). If \(e = \sqrt{v} = \sqrt{p > 2^{2^{T}}}\) and \(M M 1\) randomizes between trying to buy the asset (i.e. randomizing the bid price in \([b_{\min}; v]\) and posting the ask equal to \(a_{\alpha,1} = a_{\max}\)) and misleading \(M M 2\) by trying to sell the asset (i.e. posting a bid equal to \(b_{\min}\) and randomizing his ask in \([v; a_{\max}]\)).

Feature B) In each period \(M M 2\)'s expected payoff is zero in each of both bid and ask auctions.

**Equilibrium construction for** \(T = 2\) **and** \(p > 1\)

This subsection contains the construction of the equilibrium in the 2-periods game with \(p > 1\). We focus on the equilibrium strategies in the first trading round proceeding as follows. First, we derive some properties that MMs' quoting strategies must satisfy in an equilibrium with properties (A) and (B). Second, we provide sufficient conditions on MMs' strategies so that the resulting strategies form an equilibrium that actually satisfies these features.

A first implication of feature (A) is that the informed MM never tries to buy and sell simultaneously the asset. Therefore, \(M M 2\)'s posterior belief will depend only on his bid, if he tries to buy the asset, or on his ask, if he tries to sell it. We denote \(Post_{\text{ask}}(a)\) the \(M M 2\)'s posterior belief given that in the first period \(M M 1\) tried to sell the asset at price \(a_2 \in [v; a_{\max}]\) and posted a bid price that surely loses the bid auction, i.e. \(b = b_{\min}\). We denote symmetrically \(Post_{\text{bid}}(b)\). The posterior belief is computed using Bayes' rule.

Another implication of feature (A) is that whenever \(M M 1\) posts a bid quote that has a positive probability to win, he reveals that \(e = \sqrt{v}\). By contrast, if he posts an ask quote that has a positive probability to win the sell auction, then \(M M 2\) cannot perfectly infer the value of the asset from \(M M 1\)'s quotes.

**Lemma 3:** If an equilibrium of the game \((2; p)\) satisfies features (A) and (B), then for any ask \(a_2 \in [v; a_{\max}]\) and bid \(b_2 \in [b_{\min}; v]\), it results...
\[ G(b) = \frac{(1_i p)(V)}{p(V_i - b)} G(b) \quad (4) \]

\[ 1_i F(a) = \frac{p(V_i - a)}{(1_i p)(a_i V)} (1_i F(a)) \quad (5) \]

Furthermore, for any ask \( a \in [v; a_{\text{max}}] \) or any bid \( b \in [b_{\text{min}}; v] \), it results

\[ \text{Post bid}(b) = \frac{g(b)(V)}{(V_i V)(g(b)(V_i - b)) G(b)} \quad (6) \]

\[ \text{Post ask}(a) = \frac{f(a)(a_i V)}{(V_i V)(f(a)(a_i V) + 1_i F(a))} \quad (7) \]

Expressions (4) and (5) provide the relation between \( \text{M} M_1(V) \) and \( \text{M} M_1(V) \) quotes distribution that guarantee that \( \text{M} M_2 \)'s equilibrium payoff is zero whenever he posts an ask in \([v; a_{\text{max}}]\) and a bid in \([b_{\text{min}}; v]\). Expressions (6) and (7) provide \( \text{M} M_2 \)'s posterior beliefs after observing that \( \text{M} M_1 \) tried to buy the asset at price \( b \) or to sell it at price \( a \) respectively, given feature (A) and expressions (4) and (5).

According to feature (A), when \( \Psi = V \), in equilibrium the \( \text{M} M_1 \) never competes in the bid auction, that means \( G(b_1) = 1 \) for any \( b_1 \), \( b_{\text{min}} \). Substituting this expression in (4), it results

\[ G(b) = \frac{(1_i p)(V)}{p(V_i - b)} \quad (8) \]

Expression (8) and \( G(b) = 1 \) represent the distribution of \( \text{M} M_1 \)'s bid quotes for any bid \( b \in [b_{\text{min}}; v] \) when \( \Psi = V \) and \( \Psi = V \) respectively. Notice that substituting expression (8) and its derivative in (6) it results \( \text{Post bid}(b) = 1 \): when \( \text{M} M_2 \) observes that \( \text{M} M_1 \) tries to buy the asset in the first round, he infers that \( \Psi = V \).

Now we compute \( \text{M} M_1 \)'s global payoff when \( \Psi = V \) in an equilibrium that satisfies features (A) and (B). If in the first period \( \text{M} M_1 \) tries to buy the asset and not to sell it, then he will fully reveal that \( \Psi = V \), his continuation payoff will be zero, and his global expected payoff will be equal to his gain in the first period. Thus for any bid \( b \in [b_{\text{min}}; v] \), it results

\[ \chi_1(V; 2; p) = (V_i - b) \text{Pr}(b_{1,1} < b) + 0 \quad (9) \]
where the second term is the gain in the second period. Evaluating this expression for $b = v$ and considering that $\Pr(b_{2;1} < v) = 1$; we have that

$$\frac{1}{4}(\nabla; 2; p) = (1 \mid p)(\nabla \mid V)$$  \hspace{1cm} (10)

Substituting (10) in (9) and solving for $\Pr(b_{2;1} < b)$, it results

$$\Pr(b_{2;1} < b) = G_2(b) = \frac{(1 \mid p)(\nabla \mid V)}{\nabla \mid b}$$  \hspace{1cm} (11)

Expression (11) represents the distribution of $MM_2$’s bid quotes for any bid $b \in [b_{min}; v]$. $G_2(b)$ is such that when $V = \nabla$, $b_{2;1} \in [b_{min}; v]$ and $a_{1;1} = a_{max}$, $MM_1$’s payoff is given by expression (10).

Consider now the ask side. According to feature (A), if in the first period $MM_1$ sets an ask $a$ that has a positive probability of winning the ask auction (i.e. $a_{1;1} \in [v; a_{max}]$), then he stays out from the bid auction setting a bid $b_{2;1} = b_{min}$. Moreover such couples of bid and ask quotes belong to the equilibrium support of all $MM$ no matter their information. Thus for any $a \in [v; a_{max}]$, it results

$$\frac{1}{4}(\nabla; 2; p) = (a \mid V) \Pr(a_{2;1} > a) + (1 \mid \text{Postask}(a))(\nabla \mid V)$$  \hspace{1cm} (12)

$$\frac{1}{4}(\nabla; 2; p) = (a \mid V) \Pr(a_{2;1} > a) + \text{Postask}(a)(\nabla \mid V)$$  \hspace{1cm} (13)

Summing these two equations we have

$$\frac{1}{4}(\nabla; 2; p) + \frac{1}{4}(\nabla; 2; p) = (2a \mid \nabla \mid V)(\nabla \mid V) \Pr(a_{2;1} > a) + (\nabla \mid V)$$  \hspace{1cm} (14)

Evaluating this expression for $a = v$ and considering that $\Pr(a_{2;1} > v) = 1$ it results

$$\frac{1}{4}(\nabla; 2; p) + \frac{1}{4}(\nabla; 2; p) = 2p(\nabla \mid V)$$  \hspace{1cm} (15)

Substituting expression (15) in expression (14) and solving for $\Pr(a_{2;1} > a)$, it results

$$\Pr(a_{2;1} > a) = 1 \mid i \cdot F_2(a) = \frac{(p \mid i \cdot 1)(\nabla \mid V)}{a \mid i \cdot \frac{1}{2}(\nabla + V)}$$  \hspace{1cm} (16)

1 Remember that the equilibrium payoffs of the last stage game are:

$$\frac{1}{4}(\nabla; 1; p_t) = (1 \mid p_t)(\nabla \mid V)$$

$$\frac{1}{4}(\nabla; 1; p_t) = p_t(\nabla \mid V)$$

and, considering two stages, $p_t = \text{Postask}(a)$:
Expression (16) represents the distribution of $MM_2$'s ask quotes for any ask $a_2 \{v; a_{\text{max}}\}$ for an equilibrium that satisfies features (A) and (B).

Expressions (10) and (15) lead to

$$\frac{1}{4}(\nu; 2; p) = (3p_1 J_1(\nu; v))$$

Now we characterize the distribution of the informed market makers ask quotes.

Substituting expressions (7), in (12) and solving for $f(a)$, we obtain a first order differential equation in $F(a)$:

$$f(a) = (a - V + (\nu; a)F_2(a) - \frac{1}{4}(\nu; 2; p) J_1(a F(a))$$

Where $\frac{1}{4}(\nu)$ and $F_2(a)$ are those in expressions (10) and (16) respectively. Solving equation (18) and using the initial condition $Pr(a_1; v \epsilon V = V) = F(v) = 0$, we obtain the distribution function of the informed $MM$'s ask price when $\nu = \nu$.\footnote{The boundary condition follows from feature (A). The resulting differential equation is of the form $f(x) = -\frac{\varepsilon + \omega + x}{\varepsilon + \omega + x}(1 - F(x))$ where $\varepsilon$, $\omega$, $\varepsilon$, $\omega$, $\varepsilon$ are real numbers, and it has a closed form solution.}

We use then (5) to find $F(a)$, the distribution of $MM_1$'s ask prices when $\nu = \nu$. This method provides the distribution function of $MM$'s bid and ask quotes for any bid or ask that belong to $MM$'s equilibrium support as described in feature (A).

To complete the characterization of the equilibrium, it remains to find the values of $a_{\text{max}}$ and $b_{\text{min}}$ and to show that there are no profitable deviations.

Proposition 5: Consider the game $i (2; p)$ when $p > 1/2$ and let $F_2(a)$, $G_2(b)$, $G(b)$, $E(a)$, $\text{Post}_{\text{ask}}(\cdot)$ and $\text{Post}_{\text{bid}}(\cdot)$ be defined by expressions (16), (11), (8), (5), (7) and (6) respectively; let $F(a)$ be the solution of the differential equation (18) together with the initial condition $F(v) = 0$; let $a_{\text{max}} = \nu$ and $b_{\text{min}}$ be the solution of $G(b_{\text{min}}) = F(\nu)$.

Then the following strategies form a Bayesian equilibrium:

In the first trading round
(i) MM2 randomizes his ask and bid prices according to
\[ \text{Pr}(a_{2;1} \cdot x) = \begin{cases} 1 & \text{for } x \in [a_{\text{max}}; 1] \\ F_2(x) & \text{for } x \in [v; a_{\text{max}}[ \\ 0 & \text{for } x \in ]1; v]\end{cases} \]
\[ \text{Pr}(b_{2;1} < x) = \begin{cases} 1 & \text{for } x \in ]1; v]\end{cases} \]

(ii) If the value of the asset is $\nabla$ then, with probability $(1 - F(a_{\text{max}}))$, MM1 sets $a_{1;1} = a_{\text{max}}$ and randomizes his bid quotes on the support $[v; b_{\text{min}}]$; whereas with probability $F(a_{\text{max}})$, he sets $b_{1;1} = b_{\text{min}}$ and randomizes his ask on the support $[v; a_{\text{max}}]$. Furthermore it results
\[ \text{Pr}(a_{1;1} < x \mid \nabla = v) = \begin{cases} 1 & \text{for } x \in [1; v]\end{cases} \]
\[ \text{Pr}(b_{1;1} \cdot x \mid \nabla = v) = \begin{cases} G_2(x) & \text{for } x \in [b_{\text{min}}; v] \\ 1 & \text{for } x \in ]v; 1]\end{cases} \]

(iii) If the value of the asset is $\triangledown$, then MM1 sets $b_{1;1} = b_{\text{min}}$ and randomizes his ask on the support $[v; a_{\text{max}}]$. Furthermore it results
\[ \text{Pr}(a_{1;1} < x \mid \triangledown = v) = \begin{cases} 1 & \text{for } x \in [v; 1]\end{cases} \]
\[ \text{Pr}(b_{1;1} \cdot x \mid \triangledown = v) = \begin{cases} F_2(x) & \text{for } x \in [v; a_{\text{max}}[ \\ 0 & \text{for } x \in ]1; v]\end{cases} \]

(iv) MM2’s posterior belief is $p_2 = \text{Pr}(V = \nabla \mid a_{1;1}, b_{2;1})$ with
\[ p_2 = \begin{cases} \frac{1}{2} & \text{if } b_{1;1} > b_{\text{min}} \text{ and } a_{1;1} = \nabla \\ \text{Post}_{\text{ask}}(a_{1;1}) & \text{if } a_{1;1} < [v; a_{\text{max}}] \text{ and } b_{1;1} = b_{\text{min}} \end{cases} \]

(v) In the second trading round market makers’ strategies correspond to the equilibrium of the game $i(1; p_2)$.

(vi) Equilibrium payo$$ are $\frac{1}{2}q_2(2; p) = 0$, $\frac{1}{2}q_2(\nabla; 2; p) = (1 - p)(\nabla \mid \nabla)$ and $\frac{1}{4}q_2(\triangledown; 2; p) = (3p - 1)(\triangledown \mid \nabla)$. 

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Using the symmetry property of the game it is possible to characterize the equilibrium strategy in the first round of trade when $p < \frac{1}{2}$. In this case, in the first round, $MM_1(V)$ always tries to buy the asset, while $MM_1(V)$ randomizes between trying to buy and to sell it. The equilibrium payoffs are equal to $0$ for $MM_2$, $(2 - 3p)(V - V)$ for $MM_1(V)$ and $p(V - V)$ for $MM_1(V)$.

Finally, if $p = \frac{1}{2}$, then in the first round, all market makers set bid and ask quotes equal to $v = (V + V) = 2$ and posterior belief do not change.

We conclude this section with some remarks. First, we point out an important characteristic that is peculiar to our model: the possibility to quantify the price-leadership effect of informed market makers in quote driven markets.

Lemma 6: In the equilibrium of the game $(2; p)$ for $p > \frac{1}{2}$ an increase in $MM_1$’s ask quote in the first trading period increase $MM_2$’s expected quote in the second period, whereas $MM_2$’s first period quotes do not affect $MM_1$’s second period quotes. More precisely

\[
\frac{\mathbb{E}[a_{2,2}]}{a_{1,1}} = i \ln(p_i) \frac{(2p_i - 1)(V - V)^2}{(2a_{1,1} - 1) V} > 0
\]

\[
\frac{\mathbb{E}[b_{2,2}]}{b_{1,1}} = i \ln(1 - p_i) \frac{(2p_i - 1)(V - V)^2}{(2a_{1,1} - 1) V} > 0
\]

\[
\frac{\mathbb{E}[a_{1,2}]}{b_{2,1}} = \frac{\mathbb{E}[b_{1,2}]}{b_{2,1}} = \frac{\mathbb{E}[a_{1,2}]}{b_{2,1}} = \frac{\mathbb{E}[b_{1,2}]}{b_{2,1}} = 0
\]

Lemma 6 proves that a high ask price for $MM_1$ in the first trading round increases the expected quotes for $MM_2$ in the second round. Figure 2 plots the expected value $MM_2$’s bid and ask quotes in the second period of trade as a function of the informed $MM_1$’s ask price in the first period when $p = 0.65$ and $(V, V) = (1; 0)$.

Simulations suggest that the covariance between two successive ask quotes of $MM_1$ and $MM_2$ is roughly 15% of $V - V$ that represents a significant

\[23\] Such pure strategy equilibrium exists only for $p = \frac{1}{2}$ and it is sustained by the following out of equilibrium path belief:

\[
Pr(V = V | a_{1,1}; b_{1,1}) = \begin{cases} \frac{1}{2} & \text{for } b_{1,1} > 1/2 \\ 1 & \text{for } a_{1,1} < 1/2 \\ 0 & \text{for } a_{1,1} > 1/2 \\ \end{cases}
\]
price effect of MM1 over MM2. The effect that MM1’s first period bid price has on MM2 second period price is even sharper. Indeed, any $b_{2,1} > b_{\min}$ moves posterior belief to 1, and so in the second stage quotes jump to $\nabla$.

The second remark is that MM1 ex-interim total equilibrium payoffs for the game $(2; p)$ are continuous piecewise monotone linear function in $p$. The same happens for the equilibrium payoffs of the one shot game.

$$
\frac{1}{4} \pi (\nabla; 2; p) = \begin{cases} 
\frac{1}{2} & \text{if } p \leq \frac{1}{2} \\
 (2 \frac{3}{2} p)(\nabla; \nabla) & \text{if } p > \frac{1}{2}
\end{cases}
$$

$$
\frac{1}{4} \pi (\nabla; 2; p) = \begin{cases} 
\frac{1}{2} & \text{if } p \leq \frac{3}{2} \\
 p(\nabla; \nabla) & \text{if } p > \frac{3}{2}
\end{cases}
$$

This suggest that we can apply recursively the same method used in this section to obtain the equilibrium when the market makers interaction last an arbitrary number of periods $T$ (see the Appendix).

4 Equilibrium properties and empirical implications

In the Appendix we present the construction of an equilibrium in manipulating strategies for the $T_1$ stages game. The qualitative characteristics of such an equilibrium are the same as the ones presented in section 3, and we then refer to it for an intuitive description of the MMs’ strategies. In this section,
we compute numerical solutions of the equilibrium described in Proposition 7 using $\nabla = 1$ and $\nabla = 0$ and varying the initial belief $p$ and the length of the game $T$.

The purpose is to assess the properties in terms of informational efficiency and liquidity of our equilibrium, and to compare them to the results of Kyle, (1985) in an order-driven markets with one informed floor trader.

4.1 The value of information

Finding the value of private information has been a central issue in financial economics since the well known paradox illustrated by Grossman and Stiglitz (1976): in a competitive framework and in absence of exogenous noise, when a positive fraction of the population purchased the information, the price system is fully revealing, and so, if the information is costly, it does not pay to purchase it. Thus, the existence of equilibria where the information has a positive value seems to be related to the presence of exogenous noise in the economy. For example, in Kyle (1985) the profit of the insider trader is proportional to the volatility of noise traders’ demand. We show that this is not the case in a quote driven market, as a market maker can derive a positive profit from superior information even without exogenous noise in the market, simply behaving strategically.

In our model, the value of private information for a market maker depends on two factors. The first factor is the volatility of the fundamental, here measured by the unconditional variance of $\nabla$ that is equal to $p(1-p)(\nabla V)$. Figure 3 plots $MM_1$’s ex-ante equilibrium payoffs as a function of $p$ for the game repeated once (thin curve), 15 times, and 30 times (thick curve). The ex-ante payoff is maximum when the uncertainty in the market is high, that corresponds to $p$ close to $1/2$. Not surprisingly, private information is more valuable in markets where little is known about large shocks on the fundamentals.

The second factor is the time the $MM_1$ has to exploit his informational advantage. Figure 3 shows that the informed $MM$’s payoff increases with the number of trading rounds available before the public report occurs. However, the increment in $MM_1$’s payoff from one additional trading round decreases with $T$. Figure 4 plots $MM_1$’s ex-ante expected marginal payoff from adding two more trading rounds when $p$ is around 0.5.

To sum up, a private signal is more valuable when the volatility of the fundamentals is high, and when there are many trading rounds during which
the insider can exploit his monopoly position in information.

4.2 Informational efficiency

From the analysis of the one shot game, we know that in the long run quote driven markets are strong form efficient but they are not efficient in the short run. The characterization of the equilibrium allows to be more precise on the time required by the market to fully incorporate dealers’ private information into prices. This can be measured by the minimum number of trading rounds necessary to observe a convergence of quotes to the realized value of the asset. From the qualitative description of the equilibrium, $\text{MM}^1$’s quotes do not reveal completely his information as long as $\text{MM}^2$’s belief are sufficiently incorrect, namely as long as $e < 1 - 2^{-t}$. Thus, the minimum time required to have a strong form efficient market corresponds to the minimum time required to have $\text{MM}^2$’s belief sufficiently correct. In Figures 5 and 6 we consider a game where the public report occurs after 20 rounds of trade. These figures plot the maximum and the minimum levels that equilibrium posterior belief can reach after $t$ rounds of trade for $p = 0.07$ and $p = 0.4$ respectively. Figure 7 plots the same variables for $p = 0.4$ when there are only 10 rounds of trade before the public report.

Consider first Figure 5. When $p = 0.07$ and $\bar{V} = V$, then $e$ is high. Still, a fully revealing price will be observed between the 4-th and the 20-th round, not before. If $e$ is low (i.e. $\bar{V} = \bar{V}$), then one has to wait at least the 13-th rounds for a fully revealing price. This suggests that private infor-
information is incorporated into quotes faster when uninformed MM beliefs are correct. Comparing Figures 6 and 7, we can also see that MM1 has more incentives to quickly reveal his signal when the date of the public report is closer. Indeed the threshold $1 \cdot 2^{1/T}$ that $e$ must reach for having a positive probability to observe fully revealing quotes, decreases when the end of the game approaches.

Alternatively, we can measure the informational efficiency of the market with the evolution of the variance of the true value conditioned on all relevant public information, $\frac{\delta}{\tau}$. The closer we are to the end of the game the lower is $\frac{\delta}{\tau}$; that drops to zero when the quotes of MM1 signal his actual information. The faster the convergence of $\frac{\delta}{\tau}$ to zero, the better the properties of the market.

Figure 8 plots the expected rate of change of $\frac{\delta}{\tau}$ after each trading round for a game repeated 5 times and two different levels of the initial prior. The variance of the value of the risky asset decreases at a rate that depends on the level of the initial prior. When this prior is close to 1 or 0 (thick line), the initial variance of $\varphi$ decreases slower than when the prior is close to $1/2$ (dotted line). In both cases, however, $\frac{\delta}{\tau}$ reduces at an increasing rate, that means that less information is revealed at the early stages and MM1’s quotes reveal more during the last rounds of trade. Kyle (1985) obtains that in an order-driven market with monopoly of private information $\frac{\delta}{\tau}$ goes gradually to zero, at a constant rate. This means that our model predicts a worse performance of quote-driven markets than in order-driven markets in term of informational efficiency.
4.3 The expected cost of trading

Some empirical and experimental evidence show that the inside spreads usually widen as the moment some public announcement is supposed to be released approaches. Indeed, in Figure 9 we show that given the level of p, the expected inside spread in the first round of trade increases as the date of public report approaches. In the last stages game, the spread is maximum.

This finding is in tune with the description of equilibrium. MM1's quotes distribution change strongly with the sign of his information at the end of the game and slightly at the beginning. Thus, at the beginning of the game winner's curse is small and bid ask quotes are more concentrated around the expected value of the asset. If the market expects a value-relevant information coming soon, MM1's strategy will depend more on his private signal and competition between specialists is heavily aected by the winner's curse, that forces the uninformed to quote quite "conservatively", quoting on average high spreads. In other terms, at the end of the game more private information is released, and the winners' course effect is indeed stronger.

4.4 Price leadership

The manipulating equilibrium of proposition 6 can explain the price leadership phenomena that has been documented in the empirical literature in foreign exchange, OTC markets (Peiers). Indeed, at equilibrium there is a positive correlation between the quotes posted by the uninformed MM and the quotes that the informed MM posted in the previous trading stage. The
explanation is simple: the informed MM is more likely to post relatively high quotes when he knows $\Psi = \bar{V}$ rather than when $\Psi = V$. Thus the highest are the informed MM quotes, the more the uninformed will be induced to believe that $\Psi = \bar{V}$ and to increase on average his own quotes in the following trading stage.

Formally, for any bid price belonging to the informed MM equilibrium support, the following equation is true (see also the Appendix for the general case):

$$\frac{1}{4}(\nabla; t; p) = (\nabla; 1; b) G_2(b) + (\lambda_{h; t; 1} P \text{ast}_0(b) + \lambda_{h; t; 1})(\nabla; V)$$

Differentiating this expression with respect to $b$, and considering that $\lambda_{h; t; 1} < 0$, it results that $P \text{ast}_0(b)$ is an increasing function in $b$. A similar argument applies to the effect of the MM 1's ask price on MM 2's belief. As the uninformed MM expected quotes are increasing function of his prior belief, we can conclude that

$$\frac{\mathbb{E}[a_{2,t+1}]}{\delta_{l,t}} > 0 \quad \frac{\mathbb{E}[a_{3,t+1}]}{\delta_{l,t}} > 0$$

One should expect that this leadership effect increases as the date of the public report approaches as MM 1's quotes become more informative.

5 Conclusion

When there is asymmetric information between market makers in a quote driven market, quotes fully incorporate private information in the long run but not in the short run. Despite the highest possible transparency of the market, that allows all dealers and floor traders to observe the best informed agent's actions (i.e. his bid and ask quotes), the market is not strong-form efficient. Indeed, at equilibrium the informed market maker strategically release his private information with mixed strategies with the purpose to create some endogenous noise. This equilibrium behavior has at least four important empirical implications: first, trading prices are different from the expected value of the risky asset given market makers' information in any period; second, quotes are volatile despite there is no noise trading in the

---

Notice that the function $G_2(b)$ is continuous and piecewise differentiable with the form $\bar{\eta} \ast (\lambda_1; b)$ with $\bar{\eta} > 0$ and $\lambda > 2|\nabla; V|$. 

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market and no new shock in the fundamentals; third, there is a positive correlation between the informed market makers quotes at $t$ and the uninformed market maker quotes at $t+1$ and finally, the private information has a positive value even in such a highly transparent markets, that justifies the activity of costly collection of it by institutional dealers.

6 Appendix

Symmetry: The game $\pi(T;p)$ is symmetric with respect to the following transformation:

\[
\begin{align*}
\psi^0 &= \psi + \psi_i \psi \\
a_{i,t}^0 &= \psi + \psi_i b_{i,t} \\
b_{i,t}^0 &= \psi + \psi_i a_{i,t} \\
p^0 &= 1_i \ p
\end{align*}
\]

Proof: It is sufficient to write MMs' payoffs substituting to $a_{i,t}$ the expression $\psi + \psi_i b_{i,t}$ and to $b_{i,t}$ the expression $\psi + \psi_i a_{i,t}$; $i = 1;2$. Once MMs types are changed following (19), we obtain payoffs that differ from the original ones just for the use of the new variables $(a_{i,t}^0, b_{i,t}^0, p^0)$ and types $\psi^0$. Thus, one can derive the equilibrium of the game $\pi(T;p)$ using the equilibrium strategies of the game $\pi(T;p)$. For example if at equilibrium of the game $\pi(T;p)$ it results $\Pr(b_{i,t} \cdot x) = G(x;V)$, then there is an equilibrium of the game $\pi(T;p)$ where $\Pr(a_{i,t} > x) = \psi_i \psi_i V = G(x;V)$ and similarly for the strategies of the other players.

Proof of proposition 1: The one shot game is a first price bid-ask auction with propriety of information. The bid auction has been studied in EMW, considering that the ask auction is homomorphic to a bid auction the proposition follows from their result. For expositional completeness, we show that the described strategy profile is an equilibrium while we leave uniqueness as a consequence of EMW result.

Substituting the expression $E_\pi^a(x)$ and $G_\pi^a(x)$ in expression (3), it results that M M 2's payoff is 0 for any $b_2 \cdot v$ and any $a_2 > v$. If M M 2 sets $b_2 > v$, then he is sure to win the bid auction with an expected profit of $v - b_2 < 0$. Similarly any $a_2 < v$ would lead to a loss in the ask auction. Therefore there does not exist any profitable deviation for M M 2. Substituting the $G^a(x)$
in (1), it results that \( M M (V) \)'s payoff is equal to 
\[
(1 - p)(V - V)
\]
for any \( b_1 \cdot V \); if \( b_1 > V \), then \( M M (V) \) does not win the bid auction and his payoff is 0; if \( b_1 < V \), then \( M M (V) \) wins the bid auction and his payoff is 
\[
V - b_1 = (1 - p)(V - V) .
\]
This means that \( M M (V) \) does not have profitable deviation on the bid auction. On the ask auction any \( a_1 < V \) (resp. \( a_1 > V \)) would lead to negative profit (resp. 0 profit), so that \( a_1 = V \) is a best reply. A symmetric argument applies for \( M M (V) \).

**Proof of Theorem 2.** The proof contains one lemma.

**Lemma 3** If in equilibrium the private information is revealed with probability one at \( t \cdot T \), then time \( t \) equilibrium strategies are those of the one shot game equilibrium described in proposition 1.

**Proof:** Let \( (\theta_1(V), \theta_1(V), \theta_2) \) be some fully revealing equilibrium strategy profile that is played in \( t \). After time \( t \) there is no asymmetry of information and each player will set bid and ask prices equal to the true value of the asset. Using standard backward induction argument, it results that players' equilibrium payoff after \( t \) is equal to zero. Thus players' total equilibrium payoff from time \( t \) to \( T \) is equal to the stage \( t \) payoff.

To prove the lemma suppose that \( (\theta_4(V), \theta_4(V), \theta_2) \) is different from the unique equilibrium of the one shot game, then there is some player \( i \) (\( i = M M (V) \) or \( M M (V) \) or \( M M (V) \)) that could deviate in time \( t \) increasing his stage \( t \) payoff; furthermore he could set \( a_{i,t} = V \) and \( b_{i,t} = V \) for any \( \xi > t \) providing a continuation payoff not smaller than 0. This is a profitable deviation as it increases his time \( t \) payoff and does not decrease his continuation payoff; thus a contradiction.

Suppose that there exists an equilibrium where in some period \( t < T \) the probability of full revelation is one. Then, after time \( t \), there will be no asymmetry of information, each MM will set bid and ask prices equal to the true value of the asset and MMs will make no profits.

From Lemma 3, at time \( t \) all agents behave as if they were in the last repetition of the game whose unique equilibrium is described in section 3.1. From proposition 1, \( M M (V) \)'s equilibrium payoff is equal to \( (1 - p)(V - V) \).

Now consider the following deviation for \( M M (V) \):

\[
\begin{align*}
b_{1,t} &= V \\
a_{1,t} &= V_i
\end{align*}
\]
with ", > 0. M M 1(\text{V})’s stage \(t\) deviation payoff is equal to \(\text{Pr}(a_2 > \text{V} ; \text{V})\); this can be set arbitrarily close to 0 by choosing \(\epsilon\) small. In the one shot equilibrium the quotes \(b_{2;t} = \text{V}\) and \(a_{1;t} = \text{V} - \text{\epsilon}\) are played with positive probability only when the state of nature is \(\text{V}\); therefore when M M 2 observes \(b_{2;t} = \text{V}\) and \(a_{1;t} = \text{V} - \text{\epsilon}\), he believes that the value of the asset is \(\text{V}\) and his posterior belief in \(t+1\) will be \(p_{t+1} = 0\). Therein, in \(t+1\) the uniformed market maker will set \(a_{2;t+1} = b_{2;t+1} = \text{V}\). Thus, in \(t+1\), M M 1(\text{V}) can reach a payoff arbitrarily close to \(\text{V} - \text{\epsilon}\) by playing \(a_{1;t+1} = \text{V}\) and \(b_{1;t+1} = \text{V} + \text{\epsilon}\). It follows that M M 1(\text{V})’s overall deviation payoff can be arbitrarily close to \(\text{V} - \text{\epsilon}\) that is greater than his equilibrium payoff \((1-p_t)(\text{V} - \text{\epsilon})\), thus a contradiction.

**Proof of lemma 3:** We provide the proof for equations (5) and (7), a similar argument applies to the bid side. From feature(B), the M M 2 equilibrium payoff in the first period for the ask auction is zero, thus for any ask \(a\) belonging to M M 2’s equilibrium support, his current payoff on the ask auction is

\[
p(a, \text{V})(1_i \text{F}(a)) + (1_i \text{p}(a, \text{V})(1_i \text{F}(a)) = 0
\]

Solving for \(1_i \text{F}(a))\), it results

\[
(1_i \text{F}(a)) = \frac{p(a, \text{V})(1_i \text{F}(a))}{(1_i \text{p}(a, \text{V})(1_i \text{F}(a))}
\]

that is expression (5). Differentiating both sides with respect to \(a\), we have

\[
\frac{\partial}{\partial a} f(a) = \frac{p(a, \text{V})(1_i \text{F}(a))(\text{V} - a) \text{F}(a)}{(1_i \text{p}(a, \text{V}))^2}
\]

where \(f(a) = \text{F}(a)\). If M M 1 randomizes ask prices according to the lotteries with densities \(\text{F}(\cdot), f(\cdot)\), then it results by Bayes’ rule that

\[
\text{Pr}(\text{V} = \text{V}|a_{1:t} = a) = \frac{p\text{F}(a)}{p(a) + (1_i \text{p})f(a)}
\]

substituting (23) in this expression and simplifying, equation (7) follows.

**Proof of proposition 5:** From the construction of the equilibrium, we know that if market makers follow the strategies described in the proposition,
then their payoff are those provided in (iv). Still, we need to prove that there are no profitable deviations in the first trading stage. Consider firstly $MM_2$. If he sets $b_{2,1} \cdot v$, then his current payoff is zero. If he sets $b_{2,1} > v$, then he is sure to win the bid auction and his current expected payoff is equal to $v_i \ b_{2,1} < 0$. Thus, the uninformed $MM$ has no profitable deviation in the bid auction. A similar argument applied to the ask auction proves that $MM_2$ has no profitable deviations. Consider now $MM_1(V)$, a possible deviation is to set $b_{1,1} = b_{\text{min}} + \varepsilon_2$, $a_{1,1} = V$, $b_{2,1} = V$ and $a_{1,2} = V_i$. After observing $MM_1$’s quotes in the first stage, $MM_2$ will believe that $V = V$ and he will set $a_{2,2} = b_{2,2} = V$. Thus, $MM_1(V)$’s expected payoff from this deviation can be made arbitrarily close to

$$(V_i \ b_{\text{min}}) G_2(b_{\text{min}}) + (V_i \ V)$$

where the first term is the loss in the first period and the second term is the gain in the second period. Considering (8) it results that this expression is not greater than $\frac{1}{2}(V_2; p) = (3p_i - 1)(V_i \ V)$ if $b_{\text{min}} \ (V + V) = 2$. Another possible deviation for both $MM_1(V)$ and $MM_1(V)$ is to propose bid and ask price that have a positive probability to win both bid and ask auctions (i.e. $b_{3,1} > b_{\text{min}}$ and $a_{1,1} < V$). This is not profitable if there exist an out of equilibrium belief $\text{post}(a_2; b_2)$ such that

$$(1 \ p)(V_i \ V) + (a_{1,1} < V)(1 \ F_2(a_1)) + (V_i \ b_2) G_2(b_2) + (1 \ \text{post}(a_2; b_2))(V_i \ V)$$

$$(3p_i - 1)(V_i \ V) + (a_{1,1} < V)(1 \ F_2(a_1)) + (V_i \ b_2) G_2(b_2) + \text{post}(a_2; b_2)(V_i \ V)$$

Where $F_2(;;)$ and $G_2(;;)$ are given by (16) and (11). Easy computation shows that such a belief exists whenever $b_{\text{min}} \ (V + V) = 2$. We can conclude that if $b_{\text{min}} \ (V + V) = 2$, then $MM_1$ has no profitable deviations as cross quotes and huge spread are clearly dominated.

Finally, if $MM_1(V)$ never tries to buy and sell simultaneously the bid and the ask auction, then the probability of bidding on the ask must be equal to the probability of not bidding on the bid side this is true when $Pr(b_{2,1} = b_{\text{min}} | V) = G(b_{\text{min}}) = F(V) = Pr(a_{1,1} < V)$. Solving numerically this equation we...nd $b_{\text{min}} > (V + V) = 2$, and this complete the proof. ¥

Proof of lemma 6: Let $p_2 = Pr(V = V_i | a_{1,1}; b_{2,1} = b_{\text{inf}})$ and let $v_2 = \ldots$
\[ p_2 \nabla + (1 \_ p_2) \nabla \]

\[
E[a_{2;2}] = \int x dF^n(x) + \nabla (1 \_ F^n(\nabla)) = v_2 (1 \_ p_2) \ln (p_2) (\nabla \_ \nabla) \\
E[b_{2;2}] = \int x dG^n(x) + \nabla G^n(\nabla) = v_2 (1 \_ p_2) \ln (1 \_ p_2) (\nabla \_ \nabla)
\]

where \( F^n(\cdot) \) and \( G^n(\cdot) \) are given in proposition 1. Deriving this expression with respect to \( p_2 \) we have

\[
\frac{\partial E[a_{2;2}]}{\partial p_2} = i (\nabla \_ \nabla) \ln (p_2) > 0 \\
\frac{\partial E[b_{2;2}]}{\partial p_2} = i (\nabla \_ \nabla) \ln (1 \_ p_2) > 0
\]

Rearranging expression (13), we have

\[ p_2 = \text{Post}_{\text{ask}}(a_{1;1}) = \frac{\frac{1}{2}i(\nabla \_ \nabla; p) i (a_{1;1} \_ \nabla)}{(\nabla \_ \nabla)} (F_2(a_{1;1})) \]

Using the expression of \( F_2(a) \) provided by (16), and deriving with respect to \( a_{1;1} \) we have

\[ \frac{\partial \text{Post}_{\text{ask}}(a_{1;1})}{\partial a_{1;1}} = \frac{(2p i 1)(\nabla \_ \nabla)}{(2a_{1;1} \_ \nabla \_ \nabla)^2} \]

That is positive as \( p > 1 \Rightarrow 2 \). The result follows from \( \frac{\partial E[a_{2;2}]}{\partial a_{1;1}} = \frac{\partial E[a_{2;2}]}{\partial \text{Post}_{\text{ask}}(a_{1;1})} \) and \( \frac{\partial E[b_{2;2}]}{\partial a_{1;1}} = \frac{\partial E[b_{2;2}]}{\partial \text{Post}_{\text{ask}}(a_{1;1})} \). To prove that \( M M 1 \) quotes in the second period do not depend on \( M M 2 \)'s quotes in the first period it is sufficient to observe that the distribution of \( (a_{1;2}; b_{1;2}) \) is only affected by \( p_2 \) that does not change with \( M M 2 \)'s quotes.

6.1 The T-stages game

In this section we describe the equilibrium of the T-stages game \((p; T)\). As we focus on Markov equilibria, at each stage \( t \) of trade, players' strategy will depend only on the state of the game \( ^0_t = (T \_ t + 1; p_t) \).

To characterize the whole equilibrium bidding strategies, it is sufficient to provide the equilibrium bidding strategies profile and the equilibrium payoff...
for the ..rst round of the game \( j \) \((T; p)\) for any \( T \) and \( p \). Indeed, the MMs’ strategies in the following round will correspond to the equilibrium strategy of the ..rst round of the game \( j \) \((T \setminus 1; p_2)\), where \( p_2 = P_r(\mathbf{v} = \nabla j i, 1; b_{1,1}) \).

To begin with, we introduce the building blocks we will use to describe the equilibrium strategies. For any natural number \( t \geq 1 \) and for any natural \( j \leq t \), we define the numbers \( r_{j,t}, \overline{r}_{j,t}, \overline{\overline{r}}_{j,t}, \overline{\overline{\overline{r}}}_{j,t} \), and \( \ell_{j,t} \) as follows:

\[ r_{j,t} = \begin{cases} 0 & \text{if } j \cdot 0 \\ 1 & \text{if } j \cdot t \\ \frac{r_{j,t} + r_{j,1,t}}{2} & \text{elsewhere} \end{cases} \]

\[ \overline{r}_{j,t} = \begin{cases} \frac{r_{j,t} + r_{j,1,t}}{2} & \text{for } j \cdot t \\ 0 & \text{for } j > t \end{cases} \]

\[ \overline{\overline{r}}_{j,t} = \begin{cases} 1 & \text{for } j \cdot 0 \\ \overline{r}_{j,t} & \text{for } j \cdot t \\ \frac{(1 + \overline{\overline{r}}_{j,t - 1})r_{j,t} + (1 - \overline{\overline{r}}_{j,t - 1})r_{j,1,t}}{r_{j,t} + r_{j,1,t}} & \text{elsewhere} \end{cases} \]

\[ \overline{\overline{\overline{r}}}_{j,t} = \begin{cases} \frac{r_{j,t} + r_{j,1,t}}{2} & \text{for } j > 0 \\ 0 & \text{for } j \cdot 0 \\ \frac{r_{j,t} + r_{j,1,t}}{r_{j,t} + r_{j,1,t}} & \text{for } j \cdot t \\ \frac{(1 + \overline{\overline{\overline{r}}}_{j,t - 1})r_{j,t} + (1 - \overline{\overline{\overline{r}}}_{j,t - 1})r_{j,1,t}}{r_{j,t} + r_{j,1,t}} & \text{elsewhere} \end{cases} \]

For any state of the game we can now describe formally MMs’ equilibrium payo\( s\) and MMs’ quoting strategies during the ..rst trading stage.

Let \( i = \min \{ j \setminus r_{j,T} \geq p \} \), in other words \( i \) is such that \( p \geq r_{i,1,T} \).

Market makers equilibrium payo\( s\) are:

\[ \frac{1}{2} (\nabla; T; p) = \begin{cases} \overline{r}_{i,1,T} p + \overline{\overline{r}}_{i,1,T} (\nabla \overline{V}_i V) & \text{(24)} \\ \overline{\overline{r}}_{i,1,T} p + \overline{\overline{\overline{r}}}_{i,1,T} (\nabla \overline{V}_i V) & \text{(25)} \\ 0 & \text{(26)} \end{cases} \]
MM2’s quotes distributions are:

\[
G_2(b) = \frac{\nu_{1,T}(\nu_i V_i V) + (1 - \nu_{1,T})(\nu_i V_i V) - (1 - \nu_{1,T})^2}{\nu_{1,T} V_i (1 - \nu_{1,T}) V_i} \]

(27)

\[
F_2(a) = \frac{\nu_{1,T}(\nu_i V_i V) + (1 - \nu_{1,T})(\nu_i V_i V) - (1 - \nu_{1,T})^2}{\nu_{1,T} V_i (1 - \nu_{1,T}) V_i} \]

(28)

MM2’s posterior belief after observing that MM1 quotes a bid price equal to b and sets an ask price that surely loses the auction is given by

\[
\text{Post}_{\text{bid}}(b) = \frac{g(b)(b - V)}{(V_i V)(g(b)(b - V) - G(b))} \]

(29)

Where \( G(b) \) is MM1’s bid price distribution in equilibrium when \( \varPhi = \nabla \), and it can be obtained as the solution of the differential equation implicitly defined by the following system:

\[
\frac{1}{2} \frac{\partial}{\partial V_i} = (\nabla_i b) G_2(b) + (\nu_{1,T})_1 \text{Post}_{\text{bid}}(b) + (\nu_{1,T})_1 (\nabla_i V) \]

(30)

If \( \varPhi = \nabla \), then MM1’s bid price distribution in equilibrium is given by the following relation

\[
G(b) = \frac{p(\nabla_i b)(b - V)}{(1 - p)(b - V)} \]

(31)

MM2’s posterior belief after observing that MM1 tries to sell the asset at price a and sets a bid price that surely loses the auction is

\[
\text{Post}_{\text{ask}}(a) = \frac{f(a)(a - V)^2}{(V_i V)(f(a)(a - V) + 1 - F(a))} \]

(32)

Where \( F(a) \) is MM1’s ask price distribution in equilibrium when \( \varPhi = \nabla \), and it can be obtained as the solution of the differential equation implicitly defined by the following system:

\[
\frac{1}{2} \frac{\partial}{\partial V_i} = (a_i \nabla_i) (1 - F_2(a)) + (\nu_{1,T})_1 \text{Post}_{\text{ask}}(a) + (\nu_{1,T})_1 (\nabla_i V) \]

(33)
If $\mathcal{V} = V$, then $MM_1$'s ask price distribution in equilibrium is given by

\[
1_i \ F(a) = \frac{p(V_i \ a)}{(1_i - p)(a - V)} (1_i \ F(a))
\]

(34)

The maximum ask $a_{\text{max}}$ and the minimum bid $b_{\text{min}}$ that are posted with positive probability in the first round are the solution of the following system

\[
\frac{1}{2} F(a_{\text{max}}) = G(b_{\text{min}})
\]

(35)

Proposition 7: Consider the game $i(T; p)$. Let $i = \min_j T \cap \{T \geq p\}$ and let $F_2(\cdot), G_2(\cdot), F(\cdot), G(\cdot), \text{Post}_{\text{bid}}(\cdot), \text{Post}_{\text{ask}}(\cdot) a_{\text{max}}$ and $b_{\text{min}}$ as defined by (28), (27), (33), (30), (34), (31), (32), (29) and (35) respectively.

Then in equilibrium the bidding strategy in the first trading round are

(i) $MM_2$ randomizes his ask and bid prices according to

\[
\begin{align*}
\Pr(a_2; 1 < x) &= \begin{cases} F_2(x) & \text{for } x \in [v; a_{\text{max}}[ \\ 1 & \text{for } x \in [a_{\text{max}}; 1] \end{cases} \\
\Pr(b_2; 1 < x) &= \begin{cases} G_2(x) & \text{for } x \in [b_{\text{min}}; v] \\ 1 & \text{for } x \in [v; 1] \end{cases}
\end{align*}
\]

(ii) If the value of the asset is $V$ then, with probability $(1_i \ F(a_{\text{max}}))$, $MM_1$ sets $a_{1;1} = a_{\text{max}}$ and randomizes his bid quotes on the support $[v; b_{\text{min}}[ \); whereas, with probability $F(a_{\text{max}})$, he sets $b_{1;1} = b_{\text{min}}$ and randomizes his ask on the support $[v; a_{\text{max}}[ \); moreover it results

\[
\begin{align*}
\Pr(a_{1;1} < x[\mathcal{V} = V]) &= \begin{cases} F(x) & \text{for } x \in [v; a_{\text{max}}[ \\ 1 & \text{for } x \in [a_{\text{max}}; 1] \end{cases} \\
\Pr(b_{1;1} < x[\mathcal{V} = V]) &= \begin{cases} G(x) & \text{for } x \in [b_{\text{min}}; v] \\ 1 & \text{for } x \in [v; 1] \end{cases}
\end{align*}
\]
(iii) If the value of the asset is \( V \), then with probability \( (1 \mid E(a_{\max})) \), MM1 sets \( a_{1;1} = a_{\max} \) and randomizes his bid quotes on the support \([v; b_{\min}; v]\), whereas with probability \( E(a_{\max}) \), he sets \( b_{1;1} = b_{\min} \) and randomizes his ask on the support \([v; a_{\max}]; \) furthermore it results

\[
\begin{align*}
\Pr(a_{1;1} &< x | \varphi = V) = \begin{cases} 
0 & \text{for } x < 2^i \wedge v \wedge 1 \wedge a_{\max}; \\
(\mathbb{E}(x) + 1 &\wedge x \wedge a_{\max};) & \text{for } x \geq 2^i \wedge v \wedge 1 \wedge a_{\max}; \\
0 & \text{for } x > 2^i \wedge v \wedge 1 \wedge a_{\min}; \\
\end{cases} \\
\Pr(b_{1;1} &< x | \varphi = V) = \begin{cases} 
0 & \text{for } x < 2^i \wedge v \wedge 1 \wedge b_{\min}; \\
\mathbb{G}(x) & \text{for } x \geq 2^i \wedge v \wedge 1 \wedge b_{\min}; \\
1 & \text{for } x \geq 2^i \wedge v \wedge 1 \wedge b_{\min}; \\
\end{cases}
\end{align*}
\]

(iv) MM2’s posterior belief is

\[
p_2 = \Pr(\varphi = \varnothing | a_{1;1}; b_{1;1}) = \begin{cases} 
\mathbb{P}_{\text{ost}_{\text{bid}}}(b_{1;1}) & \text{if } b_{1;1} > b_{\min} \text{ and } a_{1;1} = a_{\max} \\
\mathbb{P}_{\text{ost}_{\text{ask}}}(a_{1;1}) & \text{if } a_{1;1} < a_{\max} \text{ and } b_{1;1} = b_{\min} 
\end{cases}
\]

(v) MMs’ equilibrium payoffs are \( \phi^*(T; p) = 0, \phi^*(V; T; p) = (\mathbb{t}_i; p + \mathbb{t}_j)(V \mid V) \) and \( \phi^*(V; T; p) = (\mathbb{t}_i; p + \mathbb{t}_j)(V \mid V) \).

Before proving the proposition we show how its statement can be related to the qualitative description of the equilibrium provided in section 3.3. Firstly we show that when \( e < 2^{i T} \), the informed MM randomizes his quotes only on the pro...table side of the market. Indeed if \( e < 2^{i T} \), then either \( \varphi = \varnothing \) and \( p < 2^{i T} \), or \( \varphi = V \) and \( p > 1 \wedge 2^{i T} \). Take the case \( \varphi = V \) and \( p > 1 \wedge 2^{i T} \), a similar argument applies to the other case. Considering that \( r_{T_1 \wedge T_1} = 1 \wedge 2^{i T} \), we have that \( i = T \) as \( p 2 \wedge r_{T_1 \wedge T_1} \), and so \( r_{i; T_1} = 1 \). Substituting such \( r_{i; T_1} \) in (27) and considering that \( \mathbb{t}^{-1}_{T; i \wedge 1} = 0 \), we have \( G_2(b) = \phi^*(V; T; p) \neq \phi^*(V \mid b) \). Substituting this expression for \( G_2(b) \) in (30), it results that the system (30) is satis...ed if and only if MM1’s continuation payo... is zero for any \( b \{b_{\min}; v\} \), i.e. \( (\mathbb{t}_i; T_1 \wedge 1 \wedge 2^{i T} \wedge \mathbb{P}_{\text{ost}_{\text{bid}}}(b) + \mathbb{t}_j; T_1 \wedge 1 \wedge 2^{i T} \wedge \mathbb{P}_{\text{ost}_{\text{ask}}}(a)) = 0 \). However, this happens if and only if MM1 fully reveals his information when he tries to buy the asset in the..rst round. This is possible if and only if in equilibrium MM1’s (V) does not try to buy the asset in the..rst round when \( p > 1 \wedge 2^{i T} \).

To see that when \( e > 2^{i T} \), MM1 participates to the unpro...table auction with positive probability consider the case \( \varphi = V \) and \( p < 1 \wedge 2^{i T} \). Then

\(^{25}\text{A similar argument applies to } \varphi = \varnothing \text{ and } p > 1 \wedge 2^{i T}.\)
Similarly when tries to buy the asset that means that only if MM's private information is not completely revealed when he tries to buy the asset that means that MM's too tries to buy the asset with positive probability.

Finally, we verify that MM's quotes fully reveal his information with positive probability if and only if e > 1 \[1 \text{ to } 2^{2i} T\], where T is the number of periods before the public report: this condition is satisfied for p > 1 \[1 \text{ to } 2^{2i} T\], in case \(\Psi = \nabla\), and for p < 2^{2i+1}, if \(\Psi = \nabla\). For instance, take T > 1 and p > 1 \[1 \text{ to } 2^{2i} T\]. If \(\Psi = \nabla\), then e = p > 1 \[1 \text{ to } 2^{2i} T\] > 2^{2i} T and therefore the informed market maker will randomize between trying to buy and trying to sell the asset. However, if \(\Psi = \nabla\) then e = 1 \[1 \text{ to } 2^{2i} T\] and MM will try to sell only. As a result, if p > 1 \[1 \text{ to } 2^{2i} T\], then MM tries to buy the asset if and only if \(\Psi = \nabla\), and so if MM observes \(b_{i+1} > b_{i} \) he infers that \(\Psi = \nabla\).26 Similarly when T > 1 and e < 1 \[1 \text{ to } 2^{2i} T\] (i.e. when 2^{2i} T < p < 1 \[1 \text{ to } 2^{2i} T\]), the probability that MM's current quotes fully reveal his information is zero. Indeed, in this case e > 2^{2i} T no matter the realization of \(\Psi\), and so MM randomizes between trying to buy and trying to sell the asset. Thus, MM cannot fully infer MM's information.

**Proof of proposition 7:** We provide here only a sketch of the proof; the complete proof is available upon request from the authors.

First, we give an intuition of the recursive construction of equilibrium supports. Fixing a date t, for all natural numbers j \cdot t we generate the numbers \(r_{j:t}\) recursively starting from \(r_{0:T} = 0\) and \(r_{1:T} = 1\). In this way, we partition the interval \([0; 1]\) in successively many j sub-intervals \([r_{j}; r_{j+1}]\) as the end of the game T gets further in time. In each of this sub-intervals, we can compute the vector \(\left(\frac{r_{j}; r_{j+1}}{r_{j}; r_{j+1}}\right)\) that gives us MM's expected payoffs p \(\in (0.1, .2)\) as described in (24) and (29). Within each sub-interval, then, MM's equilibrium payoffs are still linear in the initial p as it is the case in the one-shot game.27 This allows us to construct the equilibrium strategies exactly in the same way we construct the equilibrium for the twice repeated game. The only difference is that now the belief p follows a process that makes it jumping in different sub-intervals at each stage. Namely if p 2 \(r_{j}; r_{j+1}\) and MM tries to buy (resp. to

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26 A perfectly symmetric argument applies to the case p = 2^{2i} T.
27 Moreover, MM's equilibrium payoffs are still linear in the initial p as it is the case in the one-shot game.
sell) the asset, then posterior belief will belong to the interval $[r_{i;1:T-1}; r_{i;1:T-1}]$ (resp. $[r_{i;2:T-1}; r_{i;2:T-1}]$). Thus one has to take into account the piecewise linearity of MMs’ continuation payoffs when writing differential equations (30) and (34). Apart from this, the characterization of MMs’ equilibrium strategies is analogous to that given in section 3.3.1.

References


