Occupational Choice, Incentives and Wealth Redistributions With Scarcity of Capital

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Abstract

In a matching model of firm formation with moral hazard, we characterize the equilibrium for economies with scarcity of capital and study the effects of redistributive taxation. We give necessary and sufficient conditions determining the equilibrium matching patterns, payoffs and interest rate. These depend only on aggregate wealth and the median wealth relative to the active population, compared to setup costs and technological parameters. We confirm previous results (see [2]), showing that monotonic job specialization typically obtains when incentives are asymmetric within firms. Redistributive taxation now propagates its effects through the asset market and there may wealth nonmonotonic interest groups over median changes.

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1 Introduction

In this paper we study the interaction between occupational choices made in frictionless competitive markets, the return to capital, and wealth distribution and redistributions. We extend previous endogenous matching models of firm formation with moral hazard (e.g., Chakraborty and Citanna [2]) to allow for economies where there is an endogenous interest rate.

There are at least two reasons to study the problem of matching with an endogenous interest rate. The first relates to the political economy of redistributions. With a fixed interest rate it was found that redistributive taxes on capital had monotonic effects on class welfare (see [2], Proposition 5). Wealth classes below the median would benefit, and those above the median would lose from a redistributive policy which increases the median wealth of the population, and vice versa. This result suggests that two-party Downsian electoral competition on redistributive taxation would promote median-increasing policies of wealth redistribution. Redistributions would go in the direction of the middle classes, echoing past results by Romer [18], Roberts [17] and Meltzer and Richard [12]. However, the evidence on middle classes determining the outcome of political competition is debated (see Persson and Tabellini ([13], Ch. 7) for a survey). Is it still true that middle classes determine the direction of redistributive programs when there are repercussions on the interest rate? The second reason is purely theoretical. We want to know whether any of the results on matching patterns obtained under excess supply of capital holds true when there is scarcity of aggregate capital.

We consider economies with a large number of risk-neutral individuals identified by their wealth, or capital endowment. Individuals have access to a stochastic production technology by forming matches, or ‘firms’. Production is subject to a two-sided moral hazard problem with limited transferability of utilities. Contracts within a match specify a deterministic allocation of jobs to the parties and a distribution of the gains, taking as given what the parties may earn in other matches. Individuals and firms also have access to perfect credit markets where they can borrow or lend money at a riskfree rate.

In equilibrium, for each type, the expected gain from matching is determined endogenously to guarantee equality of demand and supply of jobs. What each individual achieves within a

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1 We will use interchangeably the terms ‘match’, ‘firm’ or ‘partnership’: in our model, they identify the same concept.

2 That is, incentive symmetric or asymmetric ‘roles’, or ‘occupations’, or ‘tasks’.
firm is set to be equal to his “outside option”, i.e., the maximum of what can be achieved in other firms or activities. Moreover, the interest rate is set to clear the credit market.

We constructively prove the existence of equilibrium. As with an exogenous interest rate, due to the limited transferability problem, monotonic job specialization typically obtains. Individuals are separated along the occupational dimension, with richer individuals choosing the occupation that is more incentive-intensive.

The general effect of scarcity of capital is to reduce the size of the productive sector. Compared to an economy with an excess supply of funds, individuals have to be richer for production to be worthwhile. The equilibrium interest rate is lower than in a first best world and equals the return from production of the match generating the lowest surplus. This surplus is determined by the wealth of the \textit{median type relative to the active population}, also determined by looking at the ratio of aggregate wealth to setup cost. Thus, the equilibrium distribution of occupations and utilities as well as the interest rate depend on the distribution of wealth.

Redistribution programs only matter for equilibrium if they affect the type equilibrium utilities, that is, if they change the median of the active population. Even though the equilibrium is constrained Pareto optimal, these policies affect aggregate surplus not only by changing the proportions of different types of firms in the economy, but also by changing the contracts, utilities and surplus attained in those firms. Trickle down effects can still be produced by policies changing the median wealth, as shown in Section 5. The interplay between the labor and the credit markets shows up in the transmission mechanism of wealth redistributions. With scarcity of capital, changes in wealth distribution affect individuals based on their position on the credit market—depending on whether they are borrowers or lenders. On the other hand, their position on the credit market depends on their occupation. Given monotonic job specialization, in general lenders can be poor and rich as well. The initial wealth distribution matters for the direction of change, and the direction of median-changing redistributive programs no longer necessarily reflects the preferences of the middle class.

Our model rests on the limited transferability assumption that arises both from the presence of ex-post limited liability constraints as well as from the absence of ex-ante transfers using random job assignment schemes or lotteries within the match. Other matching models (see, e.g., Legros and Newman [10], [11], Shimer and Smith [21], Prescott and Townsend [15]) put no restrictions on these lotteries, and assume that production is symmetric without
loss of generality. Such lotteries necessarily do not satisfy role-contingent participation con-
straints, and we study environments where these participation constraints must be satisfied, mak-
ing such lotteries useless in solving the transfer problem. These are environments where the
matching market is open until after the lottery outcome is known. Nonconvexities in job
training such as job-specific investments or intensive schooling make switching jobs costly and eli-
minate job rotations as a practical way to implement lotteries. Generally, lack of long-term
commitment due to available outside options after skills for a job are acquired break down the
use of such lotteries.

Our equilibrium notion is linked to the f-core (for a definition, see Kaneko and Wooders
[9]), and is more closely related to the one used in search models, as in Shimer and Smith [2],
for example. We adapt the definition of a search equilibrium to economies where there are
no frictions in the matching market. Another modeling approach close to ours is club theory
(see, e.g., Cole and Prescott [3], Conley and Wooders [4], Ellickson et al. [5], and Prescott and
Townsend [15]). We share their interest for describing a decentralized market mechanism. We
extend the analysis to private information economies with moral hazard team problems and
with lack of commitment. Even though we limit the convexifying effect of lotteries, we show
that the scope of competitive markets is still valid to attain constrained efficient allocations,
along the lines first sketched in the seminal paper by Prescott and Townsend [14]. Our results
on job specialization in the absence of credit market imperfections and their consequences for
redistributions are static. However, we conjecture that a dynastic version otherwise similar to
Galor and Zeira [7] (see also Banerjee and Newman [1]) can also be used as an alternative to
credit market imperfections for the creation of ‘country poverty traps’ in the macroeconomic
theory of growth and economic development.

2 The model

Economies have two dates and no aggregate uncertainty. There is only one physical commod-
ity, which is consumed at time one.

There is a continuum of individuals, of Lebesgue measure one, and indexed by $i \in [0, 1]$ (the population size is normalized to one). Individuals are risk neutral and differ in their
time zero endowment of the physical commodity, their wealth level $W_k$. Using a standard
construction, we let $M: \mathbb{R}_+ \to [0, 1]$ be a distribution function. Let $K \subset \mathbb{R}_+$ be the support
of \( \mathcal{M} \), assumed to be compact. If \( \mathcal{M} \) is continuous, let \( \overline{\pi}(k) \geq 0 \) be the density at \( k \in \mathbb{R}_+ \). If \( \mathcal{M} \) is discrete, that is, \( K \) finite, it is simply the probability of \( k \). Then \( \overline{\pi}(k) \) is interpreted as the measure of individuals of wealth level \( W_k \), with \( W_k > W_{k'} \) if \( k > k' \). So we consider both economies with a continuum or with a finite number of types. We think of \( \mathcal{M} \) as the wealth distribution of our economy, and note that \( \int_K \overline{\pi}(k) = 1 \), while \( \text{Leb}K \leq 1 \). We also denote \( i_k \) the \( i \)-th individual of wealth level or type \( k \).

At time zero, a decentralized frictionless matching market is open. Each individual has the option of engaging in a stochastic productive activity by matching with any other individual, or to remain idle or self-employed. Individuals and firms also have access to financial contracts.

Each match is made up of two partners labeled \( p \) and \( a \), respectively. We let \( d(i_k) \in D \equiv \{ \{ P_{k'} \}_{k' \in K}, \{ A_{k'} \}_{k' \in K}, I \} \) be the occupational decision of individual \( i_k \), a Lebesgue measurable function, where \( P_k \) denotes being a \( p \)-agent of any of the \( K \) types of \( a \)-agents, \( A_k \) denotes an \( a \)-agent of any of the \( K \) types of \( p \)-agents, and \( I \) corresponds to being idle. Let \( \mu : K \times D \to \mathbb{R}_+ \) be the product measurable function defined as

\[
\mu(k,d) = \text{Leb}(\{ i_k \in [0,\overline{\pi}(k)] \mid d(i_k) = d \})
\]

for each \( k \in K, d \in D \), i.e., the Lebesgue measure of individuals of type \( k \) who make the occupational decision \( d \in D \). Then, \( \mu_k(d) \equiv \mu|_{\{k\} \times D} \) gives the measure of individuals of type \( k \) who are in different occupations, and set \( \int_D \mu_k(d) = \overline{\pi}(k) \) for all \( k \), a feasibility condition. We sometimes also use the notation \( \mu_d(k) \equiv \mu|_{K \times \{d\}} \).

Upon matching, individuals decide a task allocation and how to share the stochastic output \( X \) which will be produced through the exercise of unobservable effort. Bargaining takes place within the match, while the matching market stays open. Hence, individuals are free to switch to other matches all throughout negotiations. Individuals make matching decisions based on the reservation price of their labor input, or reservation utility, which is taken as given and is assumed to be type-specific. Let \( U \equiv U(k), k \in K \) (also denoted \( U_k \)), be the reservation utility function. It acts as a price-like variable, as in other matching models (see, e.g., Shimer and Smith [21]).

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3In what follows, we adopt this sloppy but short and effective notation for this and similar Lebesgue integrals, without the infinitesimal symbol: \( \int_K \overline{\pi}(k) = \int_K \overline{\pi}(k) \, dk \).

4All statements regarding \( k \) and \( i \) can be taken to hold \( \mathcal{M} \)-a.s. and \( \text{Leb} \)-a.e., unless otherwise stated. Also note that properly speaking here \( d(i_k) \) is a measurable selection from the individual occupational choice correspondence.
We suppose that forming a firm may entail a setup cost, $C \geq 0$, which requires external financing if the combined wealth of the participants in the firm is less than $C$.

At time one, negotiations are over, production takes place, uncertainty gets resolved; output is distributed across partners, and individuals consume after meeting their financial obligations.

**Production** The production technology yields a stochastic output $X$ through the exercise of unobservable effort. We assume that effort is $e \in [0, 1]$. The cost of effort $e$ for the individual is

$$c(e) = c \frac{e^2}{2} \text{ with } c \geq 1$$

Each task entails effort, $e_p$ for the $p$–agent, and $e_a$ for the $a$–agent. Output $X$ is related to efforts as follows:

$$X = \begin{cases} X(2) & \text{with probability } f(e_p, e_a, \alpha) \\ X(1) & \text{with probability } 1 - f(e_p, e_a, \alpha) \end{cases}$$

where $X(2) > X(1) \geq 0$ and $\Delta X \equiv X(2) - X(1) < 2c$. The probability $f$ has the form:

$$f(e_p, e_a, \alpha) = \alpha e_a + (1 - \alpha) e_p$$

where $\alpha \in [1/2, 1]$ measures the degree of substitutability, and it represents a technological constraint defining the difference in occupation between being a $p$–agent or an $a$–agent.

Self-employed individuals have access to an efficient storage technology.

**Financial contracts** Perfect and complete hedging opportunities are generally impossible in the presence of moral hazard, as discussed in [8] and [16], among others, for idiosyncratic and aggregate risks, respectively. Hence we will assume that (perfect and) incomplete financial contracts exist, by assuming that there is a riskfree asset, available to both firms and individuals.\(^5\)

An amount $l$ (invested if $l > 0$, borrowed otherwise) of the riskless asset yields a payoff equal to $L(r, l) = rl$, where $r > 0$ is a gross rate of interest, taken as given by both firms and individuals.

\(^5\)We assume that in-house contracts are not observable by financial investors. Assuming the contrary allows to have complete and perfect hedging for firms, but restricted participation must be imposed by banning individuals from these contracts. This difference is not key for our result on equilibrium matching patterns.
For a self-employed individual, financial decisions boil down to whether to invest his wealth in the riskfree asset or to carry it to time one through the storage technology. For individuals forming firms, during bargaining financial investment plans \( l_F \) must be made, and ex ante financial responsibilities \( q_p \) and \( q_a \) must be assigned to the partners, with \( l_F + C \leq q_p + q_a \). For \( j = p, a \), let \( l_j \) and \( q_j \) denote the wealth agent \( j \) invests in the financial market and in the firm, respectively, with \( l_j + q_j \leq W_j \). We assume that investment in the firm can only be made if physical participation to the firm is chosen. This is in the spirit of the interpretation of firms as partnerships. Let \( W_p \) and \( W_a \) be the \( p-a \)-agent initial wealth, with \( W \equiv (W_p, W_a) \in \mathbb{R}^2 \). The financial decisions will determine \( W^F_j(r) \equiv L(r, l_j) \), the final financial wealth of the \( p-a \)-agent and the firm, respectively.

Since it is not feasible to start a firm unless individuals are able to pay back the amount borrowed regardless of the realization of output, the following condition must be met:

\[
X(1) + W^F_p(r) + W^F_a(r) + W^F_p(r) \geq 0; \quad (F1)
\]

This implies that a match must produce positive expected surplus.

**Bargaining**  The partners in a match have to specify a deterministic role allocation and a sharing rule \( w = w(X) \). Here \( w \) is the \( a \)-agent’s share, while \( W^F_p + X - w(X) \) is the \( p \)-agent’s share, by budget balance.

Individuals are assumed to be risk neutral and preferences over money and effort are assumed separable, so that

\[
\begin{align*}
  u_a(w, e_a, e_p) &= W^F_a(r) + E[w(X) | e_p, e_a] - c(e_a), \\
  u_p(w, e_a, e_p) &= W^F_p(r) + W^F_F(r) + E[X - w(X) | e_p, e_a] - c(e_p)
\end{align*}
\]

are the agents final utilities.

The sharing rule \( w \) is set to induce effort provision, subject to the incentive compatibility constraints

\[
\begin{align*}
  e_a \in \arg\max_{e \in [0,1]} u_a(w, e', e_p) \quad (IC_a) \\
  e_p \in \arg\max_{e \in [0,1]} u_p(w, e_a, e') \quad (IC_p)
\end{align*}
\]

and the limited liability constraints

\[
\begin{align*}
  w(X) + W^F_a(r) &\geq 0 \quad (LL_a) \\
  X + W^F_F(r) - w(X) + W^F_p(r) &\geq 0 \quad (LL_p)
\end{align*}
\]
Let \( \mathcal{U} = (\mathcal{U}_p, \mathcal{U}_a) \in \mathbb{R}^2 \) be the partners’ reservation utilities in the matching market. Then we let

\[
\mathbf{F}(W, r) \equiv \{(U_p, U_a) \in \mathbb{R}^2 \mid \exists w, e_a, e_p, : (LL), (IC) \text{ hold}, U_j = u_j(w, e_a, e_p) \text{ for } j = p, a\}
\]

be the feasible set of utilities given wealth levels \( W \) and the interest rate \( r \), and \( \mathbf{F}(W, \mathcal{U}, r) \equiv \mathbf{F}(W, r) \cap \{(U_p, U_a) \geq \mathcal{U}\} \). A bargaining procedure \( \mathcal{B} \) within a match at time zero is a game and a solution which picks

\[
U(\mathcal{B}, W, \mathcal{U}, r) \equiv (U_p(\mathcal{B}, W, \mathcal{U}, r), U_a(\mathcal{B}, W, \mathcal{U}, r)) \subset \mathbf{F}(W, \mathcal{U}, r)
\]

specifying a subset in the feasible set \( \mathbf{F}(W, \mathcal{U}, r) \).

In what follows, we will assume that this set is a singleton without loss of generality.

**Definition 1** A bargaining procedure \( \mathcal{B} \) is constrained efficient if for all \( W, \mathcal{U}, r \) there does not exist \( U' \in \mathbf{F}(W, \mathcal{U}, r) \) such that \( U' \geq U(\mathcal{B}, \mathcal{U}, W, r) \), with at least one strict inequality.

Constrained efficiency is valued relative to the technological, financial and informational constraints, and limited liability. The take-it-or-leave-it procedure where the \( p \)-agent makes an offer, denoted \( \mathcal{B}^* \), will be shown to be constrained efficient. In order to simplify the exposition, we keep \( \alpha \) and \( \mathcal{B} \) constant across matches (the second restriction is not binding when \( \mathcal{B} \) is constrained efficient, as shown in Proposition 1).

**Equilibrium** At time zero, individuals enter the matching and financial markets. They maximize their utility by choosing an occupation and a match, and their financial investment.

Financial choices conditional on occupations are straightforward: as there is no consumption at time zero and there is limited liability, the time zero budget constraint is

\[
l(d) = W_k - q(d), \text{ for all } d \in \mathcal{D} \text{ and all } k \in \mathcal{K}
\]

where \( q(d) \equiv 0 \text{ if } d = I \). Conditional on wanting to form a specific match, any way of splitting financial responsibilities within the firm is then seen to be equivalent up to relabeling of the sharing rule. Hence, financial investment is residual. Hereafter, we assume without loss of generality that \( q_a = 0 \text{ and } q_p = C \), so that

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\( ^6 \)When this set is empty, we take \( U(\mathcal{B}, W, \mathcal{U}, r) = rW \). This is done to avoid meaningless equilibria where the value \( U \) is not anchored to any realized match.
\[
W_a^f(r) = rW_a, W_p^f(r) = r(W_p - C) \quad \text{and} \quad W_F^f(r) = 0. \quad (1)
\]

Since asset trading is determined once occupations are, we will omit \( l(d) \) from the definition of equilibrium. Nevertheless, aggregate capital availability determines the interest rate to clear markets, and affects the individual matching opportunities.

For each individual \( i_k \), a decision \( d \in D \) yields utility

\[
U_k(d, U, r) = \begin{cases} 
  rW_k & \text{if } d = I \\
  U_a(B, W_{k'}, W_k, U_{k'}, U_k, r) & \text{if } d = A_{k'} \\
  U_p(B, W_k, W_{k'}, U_{k'}, U_k, r) & \text{if } d = P_{k'}
\end{cases}
\]

An economy will be parametrized by \( \theta = (M, \alpha, c, C, X) \) and a bargaining procedure \( B \).

An equilibrium requires optimization and consistency of individual behavior with aggregate figures. Moreover, given that the matching market is frictionless, we assume that the utilities \( U \) will be determined in such a way that there is no surplus over outside options.\(^7\)

**Definition 2** An equilibrium for an economy \( \theta \) with bargaining procedure \( B \) consists of an array \(<U, \mu, r>\) such that

(i) (optimization) for all \( k \in K \) and all \( i_k \in [0, \overline{\mu}(k)] \), given \( U \) and \( r \),

\[
d(i_k) \in \arg \max_{d \in D} U_k(d, U, r);
\]

(ii) (market clearing): for all \( k, k' \in K \)

\[
\mu_k(A_{k'}) = \mu_{k'}(P_k)
\]

and

\[
\frac{C}{2}(1 - \int_k \mu_I(k)) \leq \int_k W_k \overline{\mu}(k), \quad <0 \quad \text{only if } r = 1
\]

(iii) (no extra surplus) for all \( k \in K \), \( U(k) = \max_{d \in D} U_k(d, U, r) \).

\(^7\)Thus, unlike Rubinstein and Wolinski [20], or Gale [6], e.g., this paper is not about justifying the competitive outcome through the detailed study of bargaining. Both the existence of a finite bargaining stopping time (time one, for us) and the no extra surplus condition are assumed, and not derived as the outcome of some specific bargaining procedure.
Condition (ii) is the demand-equal-supply equivalent in this discrete goods economy, where the second equation (AM) is asset market clearing.\footnote{Conditions (i)-(iii) in Definition 2 correspond to condition (5) –the acceptance rule–, the definition of acceptance set and equation (6) –Bellman equation–, respectively, for the search equilibrium defined in Shimer and Smith [21]. Our feasibility for \( \mu(k,d) \) is their equation (1). The difference is that our outside option is not the expected value of future search, since without matching frictions search occurs today.}

Asset market clearing (AM) intuitively states that demand of capital, \( \frac{C}{2}(1 - \int K \mu_t(k)) \), where \( \frac{1}{2}(1 - \int K \mu_t(k)) \) is the number of firms formed in equilibrium, is no greater than \( \int K W_k \pi(k) \), the aggregate supply of capital. Note that \( \min(\int K W_k \pi(k)/C, 1/2) \)\footnote{With abuse of notation, hereafter \( \int K W_k \pi(k)/C = +\infty \) when \( C = 0 \).} is the maximum feasible number of firms to be formed in equilibrium.

The interest rate \( r \) is set to guarantee enough supply of funds in the economy. There is demand of funds when \( 2W_k - C < 0 \) for some \( k \) for which \( \mu_k(A_{k'}) > 0 \) and \( k' \leq k \). Then \( r \) may have to adjust to allow for self-employed individuals to provide the needed capital.

We assume that \( r \geq 1 \) since otherwise individuals would always prefer to carry wealth to the future by using the storage technology. If \( \int K W_k \pi(k) > C/2 \) then there is excess supply of capital, (AM) holds with inequality, and \( r = 1 \). In this case, the economies behave as if there was an infinite amount of capital and an exogenous interest rate, as in [2]. In this paper we concentrate on the case where there is scarcity of capital so that \( \int K W_k \pi(k) < C/2 \). The knife-edge case where \( \int K W_k \pi(k) = C/2 \) will be ignored in the characterization analysis, as there is always an equilibrium where the economy behaves as in the presence of excess supply.

We are interested in studying the equilibrium when bargaining is constrained efficient. In this case, the details of bargaining do not affect the equilibrium set, an invariance proposition. It is a consequence of part (iii) in Definition 2.

**Proposition 1** The equilibrium outcome \( \langle U, \mu, r \rangle \) is invariant to the choice of constrained efficient bargaining procedures \( \mathcal{B} \).

**Proof.** See the Appendix.

The second result we derive from the constrained efficiency of bargaining is the constrained efficiency of equilibrium.

**Constrained optimality** In order to evaluate the welfare properties of the equilibrium, we look at the utility deriving from a feasible allocation of roles to types, or a matching,
given the bargaining restrictions (in particular the incentive and limited liability constraints, and the role-contingent participation constraints). We allow for wealth transfers. A transfer policy $\tau$ is any (measurable) transformation of one economy $\theta$ into another economy $\theta_{\tau}$ with a compact (though possibly different) support of types and the same total wealth, $\int_K W_k p(k)$.\footnote{The policy $\tau$ is formally described in the proof of Proposition 2.}

The actual distribution constrains, through limited liability, the attainable utilities. However, the presence of the endogenous $r$ means that some changes in final wealth are possible through the reallocation of the riskfree asset. Instead of including directly this reallocation, we indirectly allow for such changes by imputing to each matching an interest rate as well as a utility. This is done as follows.

Recall that an economy is parameterized by $\theta = (\mathcal{M}, \alpha, C)$. Let $\mathcal{B}$, the bargaining procedure, also be given.

Although individuals produce and consume the physical commodity at time one, it is convenient to think of allocations as the number of individuals assigned to a given consumption and production plan, rather than as vectors of commodities assigned to individuals. This is consistent with other matching, or club theoretic, models (see, e.g., [3]).

**Definition 3** An allocation $\mu$ for an economy $\theta$ is a measurable function $\mu : K \times D \to \mathbb{R}_{+}$. It is feasible if for each pair $k, k'$, $\mu_k(p_k) = \mu_{k'}(A_{k'})$, and $\frac{C}{2}(1 - \int_K \mu_I(k)) \leq \int_K W_k p(k)$, while $\int_{D} \mu_k(d) = p(k)$ for all $k$. Let $\mathcal{M}_\theta$ be the set of feasible allocations.

Let $U(i)$ be the (Leb-measurable) map $[0, 1] \to \mathbb{R}$ associating a utility to each individual, in the economy $\theta$.

A feasible interest rate $r$ has the property that $r \geq 1$, and if $\mu$ is such that $\frac{C}{2}(1 - \int_K \mu_I(k)) < \int_K W_k p(k)$, then $r = 1$. This is a constraint taking into account that if feasibility holds as strict inequality, the excess supply of funds is put into the storage technology.

The utility associated to each feasible allocation $\mu$ and a feasible interest rate $r$, $U(\mu, r)$, is defined as the map

$$(i, \mu, r) \in [0, 1] \times \mathcal{M}_\theta \times \mathbb{R} \mapsto U_i(\mu, r)$$

such that for all $i, i'$, of type $k$ and $k'$, $U_i(\mu, r), U_{i'}(\mu, r) \in \mathcal{F}(W_k, W_{k'}, r)$ if $\mu(k', A_k) > 0$, and $U_i(\mu, r) = rW_k$ if $\mu(k, I) > 0$. Note that in principle the function $U(\mu, r)$ may be a correspondence, as multiple utility profiles may be compatible with a given $\mu$ and $r$. Then, to each feasible allocation $\mu$ and feasible interest rate $r$, an associated utility is $U(i) \in U(\mu, r)$.
Definition 4 For an economy $\theta$, we say that a utility profile $U(i)$ is dominated if there exist a transfer policy $\tau$, an allocation $\mu$ and an interest $r$, feasible for $\theta_r$, such that $U'(i) \in U(\mu, r)$ and $U'(i) \geq U(i)$ for almost all $i$, with strict inequality for a positive measure of $i$.

Definition 5 An equilibrium allocation $\mu$ is constrained Pareto optimal for an economy $\theta$ if its associated equilibrium utility is undominated.

We now state the optimality result.

Proposition 2 For any economy $\theta$, if the bargaining procedure $B$ is constrained efficient, the equilibrium allocation is constrained Pareto optimal.

Proof. See the Appendix.

In Section 5 we will further explore the effects of wealth redistributions on welfare. Proposition 1 allows us to focus on the take-it-or-leave-it game $B^*$ to study existence of equilibrium and equilibrium matching patterns. An equilibrium exists for all constrained efficient bargaining procedures $B$ if it exists for the take-it-or-leave-it game. Therefore, we choose the take-it-or-leave-it game $B^*$ to characterize the equilibrium outcome.

3 The take-it-or-leave-it procedure $B^*$

In the take-it-or-leave-it procedure, the $p$-agent offers the $a$-agent a sharing rule to maximize his utility. The contract is then accepted or rejected by the $a$-agent. The contract has to guarantee the $a$-agent at least a reservation utility of $U_{11}$, in expected terms (the individual rationality ($IR$) constraint).

For a given interest rate, the net (expected) surplus function and the (expected) payoffs to the $p$- and the $a$-agent in the take-it-or-leave-it bargaining procedure $B^*$ are the same as the ones obtained in [2] for an exogenous interest rate $r = 1$. The total surplus as a function of the $a$-agent’s reservation utility is strictly increasing when the $a$-agent limited liability and individual rationality constraints bind, with slope less than one. It is also ‘bell-shaped’.

Let $\Delta w \equiv w(2) - w(1)$. Let $e_p(\Delta w)$ and $e_a(\Delta w)$ be the optimal efforts derived from the ($IC$) constraints and $f(\Delta w)$ the induced probability of success, all continuous functions of $\Delta w$. Define

\[11\]In this section, $U$ is a real scalar and not a function as in Section 2. No confusion should arise.
Here \( g_p \) is the net gain to the \( p \)-agent, \( g_a \) the net gain to the \( a \)-agent, and \( g \) is the net surplus from the match, as a function of the sharing rule \( \Delta w \). It is easily checked that \( g_a \) is strictly increasing, and \( g_p \) and \( g \) are globally concave reaching interior maxima at \( \Delta w_p \) and \( \Delta w_g \) respectively, with \( \Delta w_p < \Delta w_g \) for all \( \alpha \). Furthermore, \( g_p(\Delta w_p) > g_a(\Delta w_p) \), and that \( g_p(\Delta w_g) \leq g_a(\Delta w_g) \) for all \( \alpha \), with the equality holding if and only if \( \alpha = \frac{1}{2} \). For brevity, let \( \tilde{W}_f^p(r) = r(W_p - C) + X(1) \). The \( p \)-agent thus solves:

\[
\begin{align*}
\max_{w(1), \Delta w} & \quad \tilde{W}_f^p(r) + g_p(\Delta w) - w(1) \\
\text{s.t.} & \quad rW_a + g_a(\Delta w) + w(1) \geq U \\
& \quad \min[w(1), w(1) + \Delta w] \geq -rW_a \\
& \quad \min[\Delta X - w(1) - \Delta w, -w(1)] \geq -\tilde{W}_f^p(r) \\
\end{align*}
\]

\( (C') \)

Let \( \Delta w^*(U, W, r) \) be the optimal sharing rule expressed as a function of \( U, W, r \), and let \( S(U, W, r) \equiv g(\Delta w^*(U, W, r)) \) be the optimal surplus.

**Proposition 3** Let \( \alpha \in [1/2, 1] \) be given. Assume (F1) and \( rW_a + \tilde{W}_f^p(r) + g_a(\Delta X) \geq U \). Then the optimal surplus is well-defined and has the following properties:

a) When \( U < g_a(\Delta w_p) \), then only \( (LL_a) \) binds, \( S(U, W, r) = g(\Delta w_p) \).

b) When \( g_a(\Delta w_p) \leq U < g_a(\Delta w_g) \), then \( (LL_a) \) and \( (IR) \) bind, so that \( \frac{\partial}{\partial w} S(U, W, r) \in (0, 1) \).

c) When \( g_a(\Delta w_g) \leq U \leq \tilde{W}_f^p(r) + rW_a + g_a(\Delta w_g) \), only \( (IR) \) binds, \( S(U, W, r) = g(\Delta w_g) \).

d) When \( \tilde{W}_f^p(r) + rW_a + g_a(\Delta w_g) < U \), then \( (IR) \) and \( (LL_p) \) bind, and \( \frac{\partial}{\partial w} S(U, W, r) < 0 \).

**Proof.** Omitted.\(^{12}\)

An interpretation is found in [2] for \( r = 1 \), and carries over to the general case. The optimal solution yields the following (expected) payoffs to the \( p \)-agent and the \( a \)-agent:

\[ U_a(U, W, r) = \max [U, g_a(\Delta w_p)] \] (2)

\[ U_p(U, W, r) = \bar{W}_p f(r) + rW_a + S(U, W, r) - U_a(B^*, U, W, r) \] (3)

Choosing \( U_p = U_p(U, W, r) \) and \( U_a = U \), we have that \( (U_p(U, W, r), U_a(U, W, r)) = U(B^*, W, U, r) \).

Using \( \frac{\partial}{\partial U} S(U, W, r) < 1 \) we see that \( B^* \) is constrained efficient.

\section{Characterization of equilibrium}

We characterize the equilibrium allocation as a function of \( \bar{\mathcal{M}} \), also providing a direct and constructive proof of existence of our equilibria. Essentially, we will establish conditions on the primitives which turn out to characterize the equilibrium in terms of whether \((LL_a)\) is binding for some type of \( a \)-agent.

We introduce some useful notation. For any \( k \), let

\[ P_k = \{ l \in K \mid \mu_{P_k}(l) > 0 \} \quad \text{and} \quad A_k = \{ l \in K \mid \mu_{A_k}(l) > 0 \} \]

be the sets of types who are \( p \)-agents (resp. \( a \)-agents) of type \( k \). Also, let

\[ P = \bigcup_{k \in K} P_k \quad \text{and} \quad A = \bigcup_{k \in K} A_k \]

be the set of types who are \( p \)-agents and are \( a \)-agents of some type, respectively, and \( k_0 = \inf A \), the poorest \( a \)-agent (if \( A \) is nonempty). Finally, let

\[ I = \{ k \in K \mid \mu_l(k) > 0 \} \]

be the set of self-employed types, and let \( I^c \) be the complement of \( I \), also referred to as the set of active types.\footnote{The sets \( P, A \) and \( I \) can also be obtained as inverse image of \((0, \infty)\) through \( \mu \) of appropriate measurable subsets of \( D \), and therefore are measurable.}

We use the notation \((k, j)\) for the match where \( k \) is the \( p \)-agent of \( j \). We also define \( \mu(A_k) = \int_K \mu_{A_k}(j) \), and define similarly \( \mu(P_k), \mu(A), \mu(P) \) and \( \mu(I) \). Let \( 1 = \min K \) and \( K = \max K \) without loss of generality.

As we look at economies with scarcity of capital, i.e., \( \int_K W_p \pi(k) < C/2 \), by \((AM)\) we have \( \mu(I) > 0 \). Some types have to be self-employed and provide their wealth to active (i.e.,
matched) types. If $\mu(A) = 0$ equilibrium coincides with self-employment for every type, a ‘no trade’ equilibrium. From $(AM)$ then we obtain that $r = 1$ and

$$X(1) - C + S(W_k, W_k, W_k, 1) \leq 0$$

for all $k \in K$. To rule out this case, we consider economies satisfying the following condition:

$$X(1) - C + S(W_k, W_k, W_k, 1) > 0, \text{ some } k \in K \quad (P)$$

We are interested in two matching patterns, segregation and job specialization.

**Definition 6** For an economy $\theta$, an equilibrium $<U, \mu, r>$ displays segregation if $k$ matches with $k'$ only if $k = k'$. It displays job specialization if $A \cap P$ contains at most one atomic type, and strict job specialization if $A \cap P = \emptyset$. It displays monotonic job specialization if $I \cup P = \{k \in K \mid k \leq k_0\}$ and $A = \{k \in K \mid k \geq k_0\}$.

When we have monotonic job specialization, there is a threshold type $k_0$ separating individuals into two classes, homogeneous by occupation. Rich individuals choose the occupation for which incentives are more important, while poorer individuals choose the occupation for which incentives are less important, or stay idle (i.e., they become lenders). The interest rate adjusts to make the marginal self-employed individual indifferent between taking a job and lending his wealth to others. The marginal type will be determined by asset market clearing, as a function of the whole distribution of wealth and of the setup cost. Let $r(M)$ be the equilibrium interest rate, where we stress the dependence of this rate of interest on the entire distribution $M$. It is implicitly defined by the equation

$$X(1) - r(M)C + S(r(M)W_{k_0}, W_{k_0}, W_{k_0}, r(M)) = 0 \quad (4)$$

Let

$$\tau = \frac{X(1) + g(\Delta w_g)}{C}$$

be the maximum possible value of $r$ compatible with equilibrium, and solving $(4)$. If $r > \tau$, expected net benefits from entering a firm would be negative. Throughout, we will maintain that $2rW_1 + X(1) - rC \geq 0$, all $r \in [1, \tau]$, so that condition $(F1)$ is satisfied at all matches and all $r$. Incidentally, notice how this imposes restrictions on $\theta$, and in particular on $W_1$.

---

\[^{14}\text{Whether equation (4) has a solution will be proved in Proposition 4.}\]
Using Proposition 3, the $LL_a$ binds for $k_0$ if and only if

$$U(k_0) < g_a(\Delta w_g)$$

(E*)

The preliminary fundamental properties of equilibrium can be established irrespective of the value assumed by the interest rate. Asset market clearing only plays a role by forcing self-employment in the economy for some individuals. Hence, the results obtained with an exogenous interest rate apply (see [2], Lemma 2). We summarize these properties in the following lemma, without proof.

**Lemma 1** Assume condition (P). In any equilibrium,

a) $(IR)$ binds in any match $(j,k)$ for all $j,k \in K$, so that $\mu(A) > 0$, $\mu(P) > 0$.

b) $(LL_p)$ does not bind in any match $(j,k)$.

c) For all $k \in A$, and for all $j \in I \cup P$,

$$rW_k + X(1) - rC + S(U_k,W_j,W_k,r) - U_k = U_j - rW_j = 0 \leq U_k - rW_k.$$

(5)

d) For all $k,k' \in K$, if $k > k'$ then $U_k - rW_k \geq U_{k'} - rW_{k'}$.

e) If $(E^*)$ holds, then there is monotonic job specialization.

Lemma 1.b is the key fact. If the reservation utility of a type who asks to be an $a$-agent were set high enough for $(LL_p)$ to bind, individuals would switch to less expensive partners. Hence, the wealth of the individual taking the $p$-agent job never matters in any match, realized or not. This is due to the frictionless and competitive character of the matching market. Lemma 1.b together with Proposition 3 allows us to simplify notation and write the surplus in a match $(j,k)$ as $S(U_k)$.

From Lemma 1.b, we immediately obtain that the surplus in (5) does not depend on $W_k, W_j$ or $r$. Since $U_j = rW_j$ for all $j \in I$, we must then have $U_j = rW_j$ for all $j \in I \cup P$, as Lemma 1.c states. Therefore, utility levels adjust to make the gain to a $p$-agent from hiring a wealthier individual (due to more slack in the limited liability and incentive constraints) exactly offset by an increase in payoff claimed by this individual, and the interest rate is set to equalize net payoffs of $p$-agents and self-employed types (i.e., pure lenders). Note that if $\gamma$ is defined to be the common net payoff to $p$-agents over the financial market return, $\gamma = 0$ in equilibrium: purchasing the riskfree asset or a $p$-agent has the same price in this economy. Also note that we could describe the matching market in a simplified way as a competitive market for one good, the $p$-agent job.
The equilibrium matching pattern and payoffs depend on whether or not wealth matters at all for the poorest individual taking up the $a$–agent job, as Lemma 1.e states. We now translate Lemma 1.e into conditions on the primitives. The key observation is that matches can achieve full surplus with a transfer $-w(1) = [g_a(\Delta w_g) - g_p(\Delta w_g) - (X(1) - rC)]/2$ if the net payoff is equalized across tasks, at least for one type. Therefore, for any $r$ we let

$$R(r) \equiv \{k \in K \mid rW_k \geq \frac{g_a(\Delta w_g) - g_p(\Delta w_g)}{2} - \frac{X(1) - rC}{2}\},$$

be the set of unrestricted types, and let $R^c(r)$ be its complement.

When $K = \{k\} = \{1\}$, there is only one type and equilibrium always involves segregation or self-employment. Production is an equilibrium if and only if $U = rW_1 + \frac{1}{2}[X(1) - rC + S(U)],$ and $\frac{1}{2}[1 - rC + S(U)] \geq 0$, as is obvious. Then, we let $K$ be thick, that is, contain at least two types, or $\overline{M}$ is continuous.

Letting $k_{med}$ be the median of the wealth distribution, define

$$k_* \equiv \inf\{l \in K \mid \overline{M}(l) \geq 1 - \frac{\int_K W_k \overline{\mu}(k)}{C}\}$$

as the median relative to the active population. Note that $1 - \int_K W_k \overline{\mu}(k)/C > 1/2$ and $k_* > k_{med}$ when $\int_K W_k \overline{\mu}(k) < \frac{C}{2}$. Moreover, $k_*$ depends on the entire distribution of wealth $\overline{M}$ in the economy.

In what follows, we restrict attention to economies where some conditions on the primitives hold, namely

$$X(1) - C + S(W_k, W_k, W_k, 1) > 0 \text{ for all } k \in K \quad (P^*)$$

$$\overline{M} \text{ not continuous at } k_* \Rightarrow \overline{M}(k_*) > 1 - \frac{\int_K W_k \overline{\mu}(k)}{C} \quad (ND^*)$$

Condition $(P^*)$ is a strengthening of condition $(P)$. We call productive the economies where condition $(P^*)$ holds, as then the measure of firms formed will be equal to the maximum amount feasible, given the aggregate supply of capital. Condition $(ND^*)$ is a nondegeneracy condition only used in the necessity parts of our results, in order to simplify the exposition.

\[^{15}\text{Whether such an equation has a solution will be discussed in Proposition 4. Hence, existence of equilibria for when } K = \{k\} \text{ will also be established.}\]
Proposition 4  Suppose \((P^*)\), \((ND^*)\) hold and \(K\) is thick. Then an equilibrium exists.

a) When \(k_* \in \mathbb{R}^c[\tau]\), the equilibrium is a.e. unique in utilities; the pattern of occupational choice displays monotonic job specialization and \(k_0 = k_*\), \(r = r(M) < \tau\); equilibrium utilities satisfy \(\gamma = U_{k_0} - r(M)W_{k_0} = 0\).

b) When \(k_* \in \mathbb{R}[\tau]\), \((E^*)\) does not hold; \(A \subset \mathbb{R}(\tau)\), \(\mu(P \cap \mathbb{R}^c(\tau)) > 0\) and \(\gamma = U_k - \tau W_k = 0\) for all \(k\).

c) An equilibrium displays segregation if and only if \((1/2) \int_{\mathbb{R}(\tau)} p(k) \geq \int_K W_k p(k)/C\); equilibrium utilities satisfy \(U_k - \tau W_k = 0\) for all \(k \in K\). If there is an atom at \(k = K\), then \(2W_K - C > 0\) must also be true.

Proof.  See the Appendix.

In words, Proposition 4 says that when the median relative to the active population \(k_*\) is restricted, \((LL\alpha)\) binds for type \(k_*\) and the only pattern of occupational choice consistent with equilibrium is monotonic job specialization, where the threshold level \(k_0\) is \(k_*\). This is a consequence of the fact that economies are productive, and that two jobs are needed for each firm to be formed. If \(k_*\) is unrestricted, monotonic job specialization is still consistent with equilibrium provided that there are not sufficiently many unrestricted types, but \((LL\alpha)\) does not bind for type \(k_0\).

The equilibrium profile of utilities is uniquely determined and is obtained through finding the segregation utility for \(k_0\), then setting \(\gamma = U_{k_0} - rW_{k_0}\) and using (5) as in Lemma 1.c; \(r\) is chosen to obtain \(\gamma = 0\) by solving (4).\(^{16}\)

When condition \((E^*)\) does not hold, and when there is excess supply of capital (see [2]), essentially using Lemma 1.e one shows that segregation occurs if and only if \(K = \mathbb{R}(1)\), that is, if the limited liability constraints for the \(p\)- and \(\alpha\)-agents do not bind. Here things are slightly complicated by the fact that there are self-employed types in the economy. Since asset market clearing requires self-employment in equilibrium, conditions for segregation cannot be distribution free. They depend on the proportion of unrestricted individuals relative to the aggregate number of firms (1/2 of each unrestricted type must be numerous enough to form the required number of firms).

\(^{16}\)In the knife-edge case where \((ND^*)\) is violated, the equilibrium exists, but two cases arise. If \(K\) is finite, provided that \(k_* + 1 \in \mathbb{R}^c[\tau]\), \((E^*)\) holds, and there exist a continuum of possible equilibrium profiles, lying between the unique profiles obtained by segregating \(k_*\) or \(k_* + 1\). If \(k_* + 1 \in \mathbb{R}[\tau]\), but \(k_* \in \mathbb{R}^c[\tau]\), the same equilibrium pattern will emerge, with unequal division of surplus. If \(M\) is continuous at \(k_*\), then this problem disappears.
In a segregation equilibrium the net payoff is equalized across types. It is immediate from (5) that since \( \mu(I) > 0 \), segregation implies \( X(1) - rC + S(rW_k) = 0 \) for \( k \in I^c \), and the economy is essentially at self-employment in a segregation equilibrium. The role of the assumption \( 2W_K - C > 0 \) is to rule out a knife-edge case where \( I^c = K \), \( \overline{M} \) has an atom at \( k = K \) and segregation could occur while \( (LL_a) \) is binding for the employed types, since Lemma 1.e would not bite. A distinction between segregation and monotonic job specialization is void unless there are at least two active types. Note that if the sufficient conditions in Proposition 4.c are satisfied, the equilibrium is unique in utilities, although indeterminate in matches, so that other matching configurations are compatible with equilibrium for these economies, but they are all essentially segregated. Since \( g_a(\Delta w_g) - g_p(\Delta w_g) = 0 \) if \( \alpha = 1/2 \), \( R(\pi) = K \) and all positive wealth levels are compatible with segregation, given that \( (F1) \) implies \( rW_k + (X(1) - rC)/2 > 0 \).

Proposition 4 is illustrated for the case of monotonic job specialization by the following numerical example.

![Fig. 1: Equilibrium net payoffs with scarcity of capital.](image)

In the graph, the equilibrium expected payoffs are represented for an economy with \( \alpha = 1, c = 1.1, C = X(1) = \Delta X = 1 \). There are four wealth levels, corresponding to \( W_1 = .18, W_k = .1k \) for \( k = 2, 3 \) and \( W_4 = .5 \); \( \overline{\mu} = (.14, .64, .11, .11) \), aggregate wealth is .2412, less than \( C/2 \), and equal to the size of the industrial sector, or the active population. The median relative to the active population is \( k^* = 2 \). The equilibrium interest rate \( r(\overline{M}) \) is 1.4354.
5 Effects of Redistributions

In this section, we extend the analysis of the effects on equilibrium of changes in the wealth distribution that was undertaken in Proposition 2. Proposition 4 shows that the force driving the equilibrium distribution of net payoffs is the wealth level of the median class, as with an exogenous interest rate, though now relative to the active population. When the median is affected, in general some types may benefit from the change while others may lose. Here we further characterize the welfare effects of changes in the median relative to the active population.

Let $k'_j$ denote the median wealth of the active population before ($j = 1$) and after ($j = 2$) the change; let $K^+ = \{ k \in K \mid W_k - C \geq 0 \}$, and note that $k'_1 \in K \setminus K^+$ by asset market clearing.

**Proposition 5** Let two economies $\theta^1$ and $\theta^2$ be given under Assumptions $(P^*)$ and $(ND^*)$, with $W_{k^2} < W_{k^1}$. There exists $\hat{W} \in [W_{k^2}, W_{k^1}]$ such that in equilibrium types with wealth level less than $\hat{W}$ have a (weakly) lower utility, and types with wealth greater than $\hat{W}$ have a (weakly) higher utility, unless they belong to $K^+$. Any increase in the median corresponds to the reverse effect. The effects are strict if condition $(E^*)$ holds at $k'_2$.

**Proof.** See the Appendix.

Proposition 5 allows comparisons across economies which may differ in, or have the same, aggregate wealth. When aggregate wealth is constant, changes in distribution can derive from budget balanced redistributive policies.

Proposition 5 follows from equations (4) and (5). With scarcity of capital, changes in wealth distribution yield utility changes essentially through a change in the interest rate, an asset market effect. Interest rate changes are proportional to median changes. All the lenders [resp., borrowers] are affected by a utility change [resp. inversely] proportional to the (median-induced) change in the interest rate. Monotonic job specialization implies that poorer types are always lenders, or a ‘trickle down’ effect. As the interest rate changes, some types will switch side in the asset market. For instance, if the median goes down, the interest decreases and some lenders become borrowers. Those who switch and are below $\hat{W}$ lose, the others gain.

Hence there are two interest groups in this economy, one made up of the middle class above the median, and the other by the poor and the very rich. These interest groups are nonmonotonic in wealth. Only if everyone has to borrow to form firms, i.e., $K^+$ is empty, we
have the same effects as in [2]. Proposition 5 does not immediately say how to redistribute wealth through budget balance tax reforms, i.e., what wealth classes to create through small changes in \( W_k \). Since

\[
\Delta U_k = W_k \frac{rS^k r}{1 - S^k} \Delta W_k + r \Delta W_k \quad \text{for } k \leq k^* \\
\Delta U_k = \frac{W_k - C}{1 - S_k} \frac{rS^k r}{1 - S^k} \Delta W_k + \frac{r}{1 - S_k} \Delta W_k \quad \text{for } k > k^*,
\]

(6)

if a tax reform has \( \Delta W_{k^*} > 0 \), it yields \( \Delta U_{k^*} > 0 \); if \( \Delta U_k > 0 \) for \( k \geq k^* \), then \( \Delta W_k > 0 \) for all \( k \geq k^* \) with \( k \in A \setminus K^+ \), while by budget balance \( \Delta W_k < 0 \) for some \( k < k^* \) or \( k \in K^+ \). In this case, a median increase is financed by lenders to the advantage of borrowers. Another median increasing policy could obtain welfare improvement for all those \( k \leq k^* \), by setting \( \Delta W_k = 0 \) for \( k < k^* \) and taxing the middle class just above the median. Both policies could satisfy more than 50% of the voters. However, even a median decreasing policy could also rally more than 50% of the votes, depending on the initial wealth distribution.

More particularly, consider an anonymous and general interest redistributive policy that taxes (or subsidizes) individuals at a rate \( \tau \) of their wealth, and gives them a rebate \( G \). This kind of policy shares the features of those considered in [18], [17] and [12]: although here we look at taxing wealth, the same distortionary aspects of income taxation carry over in our context. By budget balance, \( G = \tau E(W) \), so that the after tax wealth is \( \hat{W}_k = (1 - \tau)W_k + \tau E(W) \). In this case, a tax reform will have \( \Delta \hat{W}_k = E(W) - W_k \), all \( k \). It’s not difficult to see, using (6), that if \( W_{k^*} \leq E(W) \), the direction of redistribution is still dictated by the preferences of the median. However, when \( W_{k^*} > E(W) \), the median preferred outcome (\( \Delta \tau < 0 \)) may not rally 50% of the votes in Downsian political competition over tax reforms. While types with wealth \( W_k \in (E(W), W_{k^*}] \) and types with \( W_k > C \) will dislike \( \Delta \tau > 0 \), sufficiently poor individuals may benefit from taxes, as well as types with \( W_k \in (W_k, C) \). Middle classes (i.e., the median \( k^* \)) do not always dictate the distributive programs chosen by the political parties. This is again due to the capital market effects of redistributions.

Proposition 5 is illustrated by the following example. Two economies are compared, with different wealth distributions. The basic parameters of the economies (\( \alpha, c, X \) and \( C \)) are as in the economy of Fig. 1. Given these parameter values, \( S(U) = \sqrt{2U/c - U} \) in the relevant range, and the maximum surplus is .4545, while the maximum possible interest rate \( \bar{r} \) is 1.4545. Type \( k = 4 \) has now wealth \( W_4 = 1.1 \). The aggregate (mean) wealth of the economies is fixed at .3072, equal to the size of the industrial sector, or the active population. In Economy 1,
In Economy 1, $\mu_1 = (.25, .35, .30975, .09025)$, and $k^*_1 = 3$. In Economy 2, $\mu_2 = (.14, .64, .11, .11)$ and $k^*_2 = 2$. So, the median relative to the active population has decreased from Economy 1 to Economy 2. Straightforward calculations give the equilibrium utilities and interest rates in the two economies, summarized in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$U_3$</th>
<th>$U_4$</th>
<th>$r$</th>
<th>$\sum(U_k - W_k)\mu_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Economy 1</td>
<td>.26178</td>
<td>.29087</td>
<td>.43630</td>
<td>1.5999</td>
<td>1.4543</td>
<td>.13959</td>
</tr>
<tr>
<td>Economy 2</td>
<td>.25837</td>
<td>.28708</td>
<td>.44975</td>
<td>1.5980</td>
<td>1.4354</td>
<td>.13796</td>
</tr>
</tbody>
</table>

and $U_k - rW_k = 0$ for $k \leq k^*_i$, $i = 1, 2$, while $U_4 - rW_4 = .00018$ in Economy 1, and $U_3 - rW_3 = .01914$ and $U_4 - rW_4 = .01915$ in Economy 2. The last column in the previous table shows total surplus.

In the example, more inequality in Economy 2 than in Economy 1, also witnessed by an increase in the Gini coefficient from .318 to .387, means a decrease in the median, and that poorer people are worse off. Even under the assumptions of Proposition 5, there is no simple relation between wealth inequality and welfare, unless $K \leq 3$ and reducing the median implies making the wealth distribution more unequal. In general, there is no relation between median and inequality, measured through a mean preserving spread on $\mu$. A mean preserving spread may reduce or increase the median, so that raising or reducing inequality has no general clear-cut effect on the equilibrium. The same is true for the effects of the median on total surplus. This is a matter of the relative impact of the composition effect (the size of different types in the population) and of the change-in-the-median effect. However, total surplus can be increased by changing the wealth only of types $k > k_*$. This can be done by a redistributive policy that brings more equality among the $a$-agents above $k_*$, using the concavity in $W_k$ of $U_k - rW_k$ (or $U_k - W_k$) for $k > k_*$, and Rothschild and Stiglitz [19], as explained in [2].

6 Appendix

Proof of Proposition 1

Pick any constrained efficient bargaining procedure $B$ and an associated equilibrium, $< U^*, \mu^*, r^* >$. Consider an alternative constrained efficient bargaining procedure $B'$, for each pair $k', k$, starting at $U^*, r^*$. Consider the feasible set $F(W, U^*, r^*)$, with $(U^*_p, U^*_a)$ as reservation utilities. It must be that $B'$ has as utility outcome
\[(U'_{p}, U'_{a}) = U(B', W_{k'}, W_{k}, U_{k'}, U_{k}, r^{*})) \geq (U_{k'}, U_{k})\]

for all \(k, k'\) with \(\mu_{k}(A_{k'}) > 0\). Then we are done, either because \(F(W, U^{*}, r^{*})\) is empty, and \((U'_{p}, U'_{a}) = (U_{k'}, U_{k})\); or, because \((U'_{p}, U'_{a}) \in F(W, U^{*}, r^{*})\) and by constrained efficiency of \(B\). Therefore, we must have \((U'_{p}, U'_{a}) = (U_{k'}, U_{k})\). Nothing changes for pairs \((k', k)\) where \(\mu_{k}(A_{k'}) = 0\). To conclude, \(< U^{*}, \mu^{*}, r^{*} > \) must also be an equilibrium (that is, satisfies conditions (i)-(iii) in Definition 2) when the bargaining procedure is \(B'\). This shows also that the equilibrium outcome is identical for all constrained efficient bargaining procedures \(\mathcal{B}\).

Proof of Proposition 2

Let \(< U^{*}, \mu^{*}, r^{*} > \) be an equilibrium for the given economy \(\theta\), and suppose that it is not constrained optimal. Then there is a transfer policy \(\tau\), an allocation \(\mu'\) and an interest rate \(r'\), feasible for \(\theta_{\tau}\) such that \(U(\mu', r')\) dominates \(U^{*}\).

The transfer policy \(\tau\) assigns transfer \(\tau(i_{k})\) to each individual \(i_{k}\), all \(k \in K\). Since economies have a compact set of types, the image of \(\tau\) is a compact set \(T = \{\tau_{1}, \ldots, \tau_{J}\}\). Let \(\mu(k, \tau)\) be the measure of individuals \(i_{k}\) receiving transfer \(\tau \in T\), where \(\int_{T} \mu(k, \tau) = \mu(k)\). For any individual \(i_{k}\), let \(W_{k, \tau} = W_{k} + \tau\) be \(i_{k}\)'s wealth level after transfer \(\tau\) is performed. Transfers must then satisfy \(\tau(i_{k}) \geq -W_{k}\) all \(i_{k}\), all \(k\), and \(\int_{K} \int_{T} \tau \mu(k, \tau) = 0\), where the equality states that total wealth is unchanged.

If \(i_{k}\) is self-employed under \(\mu'\), then \(r'W_{k, \tau} \geq r^{*}W_{k}\), with strict inequality if \(U_{i_{k}}(\mu', r') > U_{k}^{*}\). For any match \((i_{k}, i_{k'})\) under \(\mu'\), where \(i_{k}\) and \(i_{k'}\) have received transfers \(\tau\) and \(\tau'\) respectively, we must have \(r^{*}(W_{k} + W_{k'} - C) \leq r'(W_{k, \tau} + W_{k', \tau'} - C)\). To see this, recall from our discussion of expression (1) that, up to relabeling, only the total wealth of the firm matters for contracts. Suppose that the previous inequality does not hold. Then

\[(U_{i_{k}}(\mu', r'), U_{i_{k'}}(\mu', r')) \in F(W_{k, \tau}, W_{k', \tau'}, U_{k}, U_{k'}, r') \subset F(W_{k}, W_{k'}, U_{k}, U_{k'}, r^{*}),\]

where both sets are nonempty, or we are done. Let \(w', e'_{a}, e'_{p}\) be the contract that generates \(U_{i_{k}}(\mu', r'), U_{i_{k'}}(\mu', r')\). Using the same contract, from the definition of \(u_{a}, u_{p}\) and again relabeling if necessary, there exists \(\hat{U} \in F(W_{k}, W_{k'}, U_{k}, U_{k'}, r^{*})\) with \(\hat{U}_{p} > U_{i_{k}}(\mu', r') \geq U_{k}\) and \(\hat{U}_{a} \geq U_{i_{k'}}(\mu', r') \geq U_{k'}^{*}\), a contradiction with the constrained efficiency of \(B\). Following the same logic, if \(U_{i_{k}}(\mu', r') > U_{k}^{*}\) or \(U_{i_{k'}}(\mu', r') > U_{k'}^{*}\), then \(r^{*}(W_{k} + W_{k'} - C) < r'(W_{k, \tau} + W_{k', \tau'} - C)\).

Then, using the feasibility of \(\mu'\), the definition of \(\mu(k, \tau)\), and obvious notation, and summing over all self–employed individuals and all matches, we obtain
\[ r'(\int_K \int_T W_{k,\tau} P(k, \tau) - C \frac{1}{2} \int_k \int_T \mu'_I(k, \tau)) = \int_K \int_T r'W_{k,\tau} P(k, \tau) + \]
\[ \int_K \int_T \{ \int_K \int_T r'(W_{k,\tau} + W_{k',\tau'} - C) \mu'(k, \tau, P_{k',\tau'}) \} > \int_K \int_T r^*W_{k,\tau} P(k, \tau) + \]
\[ \int_K \int_T \{ \int_K \int_T r^*(W_k + W_{k' - C}) \mu'(k, \tau, P_{k',\tau'}) \} = r^*(\int_K W_{k,\tau} P(k) - C \frac{1}{2} \int_K \mu_I(k)) \]

If \[ 1 - \frac{1}{2} \int_K \mu'_I(k, \tau) = \frac{1}{2} \int_K \frac{\int_T W_{k,\tau} P(k, \tau)}{C} \], we have a contradiction, using (AM). If not, \( r' = 1 \), then \( r^* \geq 1 \), and \((r^* - 1)W_k \leq \tau \) if \( i_k \) is self-employed under \( \mu' \) and receives \( \tau \), while \( r^* (W_k + W_{k' - C}) \leq \tau' + \tau'' \) if \((i_k, i_{k'})\) match under \( \mu' \), and receive transfers \( \tau', \tau'' \), again with strict inequalities if utilities are strictly higher than in equilibrium. But then \( \int_T \tau > 0 \), while \( \int_K \int_T \tau P(k, \tau) = 0 \), a contradiction. □

**Proof of Proposition 4**

Since \( \int_K W_{k,\tau} P(k) < \frac{C}{2} \), from (AM) we obtain that \( r > 1 \). For if \( r = 1 \), then by condition (P*), any \( j \in I \) would strictly prefer matching with its own type as a p-agent to self-employment. Therefore, in this case (AM) must hold with equality.

(a) We show that \( k_* \in \mathbb{R}^c[7] \) implies \((E^*)\). If not, an equilibrium where \((E^*)\) does not hold must obtain, and then \( U_k \geq g_a(\Delta w_g) \) for all \( k \in A \), so that from (5), we obtain that there exist constants \( \gamma_a, \gamma \geq 0 \) such that

\[ U_k - rW_k = \gamma_a \text{ for all } k \in A \]
\[ U_j - rW_j = \gamma \text{ for all } j \in P \]

with \( \gamma_a \geq \gamma \) and

\[ \gamma_a + \gamma = X(1) - rC + g(\Delta w_g) \geq 0 \]

Notice that \( \gamma = 0 \) since \( \mu(I) > 0 \). If \( \gamma_a > \gamma \), then \( A \cap P = \emptyset \). By market clearing (AM), \( k_0 \leq k_* \). If \( k_0 < k_* \), using Lemma 1.d, \( k \in A \) for all \( k \geq k_0 \), again contradicting market clearing; otherwise, by \((ND^*)\) and (AM), \( \mu(A \cap P) > 0 \), hence \( A \cap P \neq \emptyset \), a contradiction. If \( \gamma_a = \gamma = 0 \), then again we contradict \( k_* \in \mathbb{R}^c[7] \).

Now, by Lemma 1.e, (AM) is

\[ \int_k \mu_{k_0}(A_j) + (1 - M_0(k_0)) = \int_k \int_k \mu(k, A_j) = \int_k W_{k,\tau} P(k) \]

Since \( M(k_*) > 1 - \frac{1}{2} \int_k W_{k,\tau} P(k) \) using \((ND^*)\), we must have \( k_0 = k_* \). For suppose \( k_0 > k_* \). Then \( \int_{k \geq k_0} \int_k \mu(k, A_j) < \frac{1}{2} \int_k W_{k,\tau} P(k) \) a contradiction. Similarly, \( k_0 < k_* \) is inconsistent with market clearing.
Then $\int_{\mathbf{K}} \mu_{k_0}(A_j) > 0$ and $\min\{\int_{\mathbf{K}} \mu_{k_0}(P_j), \mu_{k_0}(I)\} > 0$. From (5) in Lemma 1.c, we must have for all $j < k_0$ and all $k > k_0$,

$$U_j - rW_j = U_{k_0} - rW_{k_0} = \gamma = 0$$

(1)

$$0 = rW_k + X(1) - rC + S(U_k, W_k, r) - U_k$$

(2)

Since $k_0 = k^*$, using (7.1) in (7.2) when $k = j = k^*$, we obtain equation (4) as

$$X(1) - rC + S(rW_{k^*}, W_{k^*}, W_{k^*}, r) = 0$$

(8)

Hence, $r = r(\mathbf{M})$ and from (E*) and (7.1) we must then have $r(\mathbf{M})W_{k^*} < g_a(\Delta w_g)$. Notice that this is consistent with $k^* \in \mathbf{R}^e[\mathbf{T}]$. For if not, using the definition of $\mathbf{R}[\mathbf{T}]$ we would have $rW_{k^*} \geq g_a(\Delta w_g)$. Then using $rW_{k^*}$ as utility level, $r(\mathbf{M}) = \mathbf{T}$, violating $r(\mathbf{M})W_{k^*} < g_a(\Delta w_g)$.

To show that (8) has a solution, let

$$\psi(r) \equiv X(1) - rC + S(rW_{k^*}, W_{k^*}, W_{k^*}, r)$$

We observe that $\psi$ is continuous, $\psi(\mathbf{T}) \leq 0$ by (F1), positive expected maximum surplus and the definition of $\mathbf{T}$. Also, $\psi(1) > 0$ by property (P*). By the Intermediate Value Theorem a solution to (7.2) when $k = j = k^*$ exists for $r$ in the range $(1, \mathbf{T}]$, and is unique by the properties of the surplus function as in Proposition 3. Hence $r(\mathbf{M})$ is well defined if $k_0 = k^*$.

Consider system (7) when $k_0 = k^*$. That is, letting $U_k = rW_k$ for all $k \leq k^*$, equation (7.2) when $k = j = k^*$ is solved for $r$ by $r(\mathbf{M}) < \mathbf{T}$. Given a solution to (7.2) when $k = j = k^*$, a solution to (7.1) and (7.2) also exists for all $k > k^*$ and all $j < k^*$ using the same logic as to solve (4), and is unique by Proposition 3.

The profile of utilities and $r$ so obtained is an equilibrium with $\mathbf{I} \cup \mathbf{P} = \{k \in \mathbf{K}|k \leq k^*\}, \mathbf{A} = \{k \in \mathbf{K}|k \geq k^*\}$ as a feasible allocation $\mu$ can be found. If $\mathbf{M}$ is not continuous, we match types (for instance, but not necessarily) in a positive assortative way, until all $k > k^*$, in measure $1 - M(k^*) \leq \int_{\mathbf{K}} W_k \mathbf{P}(k)/C$, are assigned as $a$-agents to types below $k^*$. The process will stop in finite time. We then choose $\mu_{k^*}(A_{k^*}), \mu_{k^*}(I)$ to satisfy feasibility for $k^*$ and asset market clearing (AM) (two equations in two unknowns). Notice that (AM) is the equation $\varphi(\mu_{k^*}(I)) = 0$, where

$$\varphi(\mu_{k^*}(I)) = \int_{\mathbf{K}} W_k \mathbf{P}(k) - (C/2)(1 - \mu_{k^*}(I)).$$

Clearly $\varphi$ is continuous, $\varphi(0) < 0$ and $\varphi(\mathbf{P}(k^*)) > 0$, so that by the Intermediate Value Theorem, there is $\mu_{k^*}(I) \in (0, \mathbf{P}(k^*))$ with $\varphi(\mu_{k^*}(I)) = 0$. Now we choose $\mu_{k^*}(A_{k^*})$ to satisfy feasibility for $k^*$.
Otherwise, let \( \mu_I|_{K_*} \) and \( \overline{\mu}|_{K_*} \) be \( \mu_I(k) \) and \( \overline{\mu}(k) \) restricted to \( K_* = \{ k \in K | k \leq k_* \} \) as domain. (AM) is the equation \( \overline{\varphi}(\mu_I|_{K_*}) = 0 \), where

\[
\overline{\varphi}(\mu_I|_{K_*}) = \int_K W_k \overline{\mu}(k) - (C/2)(1 - \int_{K_*} \mu_I(k)).
\]

Clearly again, \( \overline{\varphi}(0) < 0 \) and \( \overline{\varphi}(\overline{\mu}|_{K_*}) > 0 \). Consider the subspace \( C(\mathbb{R}(\tau), [0, 1]) \) of continuous functions \( 0 \leq \mu_I|_{K_*} \leq \overline{\mu}|_{K_*} \) with any topology for which summation and scalar multiplication of these functions are continuous. Since the convex combination of any two elements of this space is still in the space, it is connected, and the function \( \overline{\varphi} \) continuous as it is the Lebesgue integral of \( \mu_I|_{K_*} \). Again by the Intermediate Value Theorem, there exists \( \mu_I|_{K_*} \) such that the asset market clears. Then we proceed by assigning \( \int_{k \leq k_*} \mu_k(A_j) = \overline{\mu}(k) \) for \( k > k_* \) to the measure \( \overline{\varphi}(k_*) - \int_{K_*} \mu_I(k) \) of types \( k \leq k_* \).

(b) Suppose that \( k_* \in \mathbb{R}[\tau] \). If \( 0 \leq k_0 \in \mathbb{R}[\tau] \), then \( k_0 < k_* \), and \((E^*)\) holds, but using part (a), \( k_* \in \mathbb{R}[\tau] \), a contradiction. So \( k_0 \in \mathbb{R}[\tau] \), and \( A \subset \mathbb{R}[\tau] \). Setting \( r = \tau \) and \( U_k = \tau W_k \) for all \( k \), equation (4) is solved and \((E^*)\) does not hold, and \( \gamma = 0 \). Now, (AM) and \( A \subset \mathbb{R}[\tau] \) imply \((C/2) \int_{\mathbb{R}(\tau)} \overline{\mu}(k) < \int_k W_k \overline{\mu}(k) \), so that \( \mu_k(I) < \overline{\mu}(k) \) for some \( k \in \mathbb{R}[\tau] \), and \( \mathcal{P} \cap \mathbb{R}[\tau] \neq \emptyset \).

(c) First, we show necessity. For segregation to be an equilibrium, given Lemma 1.b and from (5) in Lemma 1.c, we have

\[
U_{k'} - rW_{k'} = U_k - rW_k = \gamma = [X(1) - rC + S(U_k)]/2 \quad \text{for all } k, k' \in K. \tag{9}
\]

and \( X(1) - rC + S(rW_k, W_k, r) \leq 0 \) if \( k \in I \). Obviously, for the self-employed types \( k \in I \), we have \( U_k = rW_k \), so that \( r = \tau \) and \( \gamma = 0 \).

Next, we show that in any segregation equilibrium the \((LL_a)\) cannot bind.

First, assume that \( I^c \) is a single atom, and suppose it does. Since \( I^c = \{ k_0 \} \), then \( X(1) - rC + S(rW_{k_0}) = 0 \) and \( r < \tau \), while \( X(1) - rC + S(rW_k, W_k, r) \leq 0 \) for \( k \in I \). If \( k > k_0 \) the \((LL_p)\) must bind in a \((k, k)\) match; for if not, \( X(1) - rC + S(rW_k) > X(1) - rC + S(rW_{k_0}) = 0 \) and the \( p \)-agent in such a match gets \( rW_k + X(1) - rC + S(rW_k) > U_k \), a contradiction. But as Lemma 1.b established that \((LL_p)\) cannot bind in any match of any equilibrium, it must be that \( k_0 = K \), \( \overline{\varphi} \) has an atom at \( K \), and by assumption \( 2W_K - C > 0 \). However, asset market clearing (AM) is

\[
\int_{k < K} W_k \overline{\mu}(k) + W_K \mu_K(I) + (2W_K - C)(\overline{\mu}(K) - \mu_K(I))/2 = 0
\]

implying \( 2W_K - C < 0 \), a contradiction establishing that the \((LL_a)\) cannot bind at any segregation equilibrium with \( I^c = \{ K \} \).
When $\mathbf{I}^c$ is thick, in any segregation equilibrium the $(LLa)$ cannot bind, simply as a converse to Lemma 1.e.

Since firms operate at full surplus $g(\Delta w_g)$ and $r = \overline{r}$, $\mathbf{A} \subset \mathbf{R}(\varphi)$ follows immediately. Moreover, we have that $\mu(k, I) = \overline{\mu}(k)$ for $k \in \mathbf{R}(\varphi)$. To see this, observe that given $r, U_k = rW_k$ and any $k \in \mathbf{R}(\varphi)$ will be indifferent between forming a firm with $k' \in \mathbf{R}(\varphi)$ or staying self-employed. Being $p$-agent of any $k' \in \mathbf{R}(\varphi)^c$ cannot yield $g$, hence any $k \in \mathbf{R}(\varphi)$ would get an expected net payoff of $X(1) - \varphi C + S(rW_k, W_k, W_k', \varphi) < 0$ for $k' \in \mathbf{R}(\varphi)^c$. The same reasoning shows that any $k \in \mathbf{R}(\varphi)^c$ can either be self-employed or forming a firm with any $k' \in \mathbf{R}(\varphi)$. By segregation, $\mu_I(k) = \overline{\mu}(k)$ for $k \in \mathbf{R}(\varphi)^c$. As in part (a), let $\mu_I|_{\mathbf{R}(\varphi)}$ and $\overline{\mu}|_{\mathbf{R}(\varphi)}$ be $\mu_I(k)$ and $\overline{\mu}(k)$ restricted to $\mathbf{R}(\varphi)$ as domain. Then let

$$\widetilde{\varphi}(\mu_I|_{\mathbf{R}(\varphi)}) = \int_K W_k \overline{\mu}(k) - (C/2) \int_{\mathbf{R}(\varphi)} (\overline{\mu}(k) - \mu_I(k)).$$

Observe that (AM) requires $\widetilde{\varphi}(\mu_I|_{\mathbf{R}(\varphi)}) = 0$ for some $\mu_I|_{\mathbf{R}(\varphi)}$; also, $\widetilde{\varphi}$ is nondecreasing in $\mu_I|_{\mathbf{R}(\varphi)}$. If $(C/2) \int_{\mathbf{R}(\varphi)} \overline{\mu} < \int_K W_k \overline{\mu}(k)$, then $\widetilde{\varphi}(0) > 0$, and a fortiori $\widetilde{\varphi}(\mu_I|_{\mathbf{R}(\varphi)}) > 0$ for all $\mu_I|_{\mathbf{R}(\varphi)} > 0$ on a positive measure of $k \in \mathbf{R}(\varphi)$, contradicting (AM). Therefore, $(C/2) \int_{\mathbf{R}(\varphi)} \overline{\mu}(k) \geq \int_K W_k \overline{\mu}(k)$.

For sufficiency, we construct a segregation equilibrium as follows: given $r = \overline{r}$, $k \in \mathbf{R}(\varphi)$ implies that these types can form wealth-homogeneous firms at maximum surplus. Let $U_k = rW_k$ for all $k \in \mathbf{I}$. As we showed for necessity, at $r, U$, segregation is consistent with optimization, and then we have $\mu(k, I) = \overline{\mu}(k)$ for $k \in \mathbf{R}(\varphi)^c$, while $\mathbf{I}^c \subset \mathbf{R}(\varphi)$. The only issue remaining is asset market clearing, determining the number of firms formed in equilibrium. We need to find a solution to $\widetilde{\varphi}(\mu_I|_{\mathbf{R}(\varphi)}) = 0$, one equation in the unknown function $\mu_I|_{\mathbf{R}(\varphi)}$. But if $(C/2) \int_{\mathbf{R}(\varphi)} \overline{\mu}(k) \geq \int_K W_k \overline{\mu}(k)$, then $\widetilde{\varphi}(0) \leq 0$, while $\widetilde{\varphi}(\overline{\mu}|_{\mathbf{R}(\varphi)}) > 0$. If $\mathbf{K}$ is finite, $\mu_I|_{\mathbf{R}(\varphi)}, \overline{\mu}|_{\mathbf{R}(\varphi)}$ are vectors; now, $\widetilde{\varphi}$ is continuous and $[0, \overline{\mu}|_{\mathbf{R}(\varphi)}]$ is connected, so by the Intermediate Value Theorem, there exists $\mu_I|_{\mathbf{R}(\varphi)}$ such that the asset market clears. When $\overline{\mathbf{M}}$ is continuous, we consider the subspace $C(\mathbf{R}(\varphi), [0, 1])$ of continuous functions $0 \leq \mu_I|_{\mathbf{R}(\varphi)} \leq \overline{\mu}|_{\mathbf{R}(\varphi)}$ and proceed with the Intermediate Value Theorem as in part (a). Then segregation is an equilibrium with $\mathbf{A} \subset \mathbf{R}(\varphi)$ nonempty.

**Proof of Proposition 5**

a) Consider two economies $\theta^1$ and $\theta^2$, with medians relative to the active population $W_{k_1}$ and $W_{k_2}$, and $W_{k_2} < W_{k_1}$, while there is aggregate scarcity of capital, and $\alpha, C$ are the same. Denote by $r^1, r^2$ and $U^1, U^2$ the corresponding equilibrium interest rates and utility vectors. Equation (4) implicitly defines $r(\overline{\mathbf{M}}) = r(W_{k^*})$, and

$$r'(W_{k^*}) = \frac{rS'_*}{C - W_{k^*}S'_*} \geq 0 \quad (10)$$
since $C - W_k > 0$ (we let $S'_s \equiv S'(rW_k)$). Then, $r'(W) \geq 0$ for $W \in [W_{k2}, W_{k1}]$, and $r^2 \leq r^1$. Since $\gamma^1 = \gamma^2 = 0$, $U^2_k \leq U^1_k$ for $W_k \leq W_{k2}$. As for $W_k > W_{k2}$, letting $S'_k \equiv S'(U_k)$, if $W_k \geq W_{k1}$ equation (5) implicitly defines $U_k(r)$ such that $U'_k(r) = \frac{W_k - C}{1 - S'_k}$ (whether or not $k$ is in the support of both $\theta^1$ and $\theta^2$). If $W_k$ is such that $k \in A^1 \backslash K^+$, $W_k - C < 0$, and $U^2_k \geq U^1_k$, and notice that $W_{k2}$ is in this set. If $W_k$ is such that $k \in K^+$, and notice that $W_k > W_{k1}$, we have that $U^2_k \leq U^1_k$. If $W_k \in [W_{k2}, W_{k1}]$, the difference $U^2_k - U^1_k$ is given by $\phi(W_k, r^2) - r^1 W$, where $\phi(W_k, r^2)$ is the solution to (5). At $W_k = W_{k2}$, $U^2_k - U^1_k \leq 0$, while at $W_k = W_{k1}$ the difference is nonnegative. Hence, the Intermediate Value Theorem gives the existence of a $W \in [W_{k2}, W_{k1}]$, as required. Strict inequalities hold if $k^*_2 \in R^c(\tau)$. 

References


