Abstract

We propose a clientele-based theory of the optimal maturity structure of government debt. We assume a three-period economy in which clienteles correspond to generations of agents consuming in different periods. An optimal maturity structure exists even in the absence of distortionary taxes, and consists in the government replicating the actions of private agents not yet present in the market. The optimal fraction of long-term debt increases in the weight of the long-horizon clientele, provided that agents are more risk-averse than log. We examine how changes in maturity structure affect equilibrium prices and show that in contrast to most representative-agent models, lengthening the maturity structure raises the slope of the yield curve.
1 Introduction

The government-bond market involves many distinct investor clienteles. For example, pension funds and insurance companies invest typically in long maturities as a way to hedge their long-term liabilities, while asset managers and banks’ treasury departments hold shorter maturities. Clienteles’ demands vary over time in response to demographic or regulatory changes. This time-variation can have important effects both on the yield curve and on the government’s debt-issuance policy.

The UK pension reform provides a stark illustration of clientele effects. Starting in 2005, pension funds were required to mark their liabilities to market, discounting them at the rates of long-maturity bonds. This raised their hedging demand and had a dramatic impact on the yield curve. For example, in January 2006 the inflation-indexed bond maturing in 2011 was yielding 1.5%, while the 2055 bond was yielding only 0.6%. The 0.6% yield is very low relative to the 3% historical average of UK long real rates. Moreover, the downward-sloping yield curve is hard to attribute to an expectation of rates dropping below 0.6% after 2011. The steep decline in long rates induced the UK Treasury to tilt debt issuance towards long maturities. For example, bonds with maturities of fifteen years or longer constitute 58% of issuance during financial year 2006-7, compared with an average of 40% over the four previous years.\footnote{The issuance numbers are from the website of the UK Debt Management Office. Another illustration of clientele effects is the French Treasury’s first-time issuance of a 50-year bond in 2005, in response to strong demand by pension funds.}

While clientele considerations seem to influence the practice of government debt issuance, they are largely absent from the theory. The first challenge for normative theories of government debt is to overcome the Ricardian equivalence result of Barro (1974): in a representative-agent model with non-distortionary taxes, the level and composition of government debt are irrelevant. To get around the irrelevance result, the literature has emphasized the distortionary aspect of taxes. The level and composition of government debt then become a tool for the government to smooth taxes across states and over time, and so to raise welfare. However, the literature assumes a representative agent, thus precluding clienteles.

In this paper we propose a theory of optimal maturity structure that emphasizes the role of clienteles. We consider an economy in which clienteles correspond to generations of agents consuming in different periods. Because future generations cannot trade before they are born, markets are incomplete and intergenerational risk-sharing is imperfect. We show that the government can improve risk-sharing through its debt-issuance policy, and we derive an optimal maturity structure in the absence of distortionary taxes. Under plausible parameter values, the optimal maturity structure depends on the mix of generations in an intuitive manner, involving more long-term debt when
the generation with the longer lifespan is wealthier. We also examine how changes in the maturity structure of government debt or in the mix of clienteles affect the yield curve. For instance, we show that consistent with practical intuition but in contrast to most representative-agent models, lengthening the maturity structure raises the slope of the yield curve.

We conduct most of our analysis in a three-period setting, described in Section 2. There are three agents with identical CRRA preferences. In the initial period $t = 0$, only agents 1 and 2 are alive and they receive an exogenous endowment. They invest their endowment until the time when they need to consume, which is period 1 for agent 1 and period 2 for agent 2. Agent 3 is born in period 1, receives endowments in periods 1 and 2, and consumes in period 2. One interpretation of this setting is that agents 1 and 2 represent different generations currently alive, while agent 3 represents an aggregate of all future generations. All agents can invest in a one-period linear production technology. The return on this technology is riskless and pins down the one-period interest rate. The return between periods 1 and 2 becomes known only in period 1. The government incurs an expenditure in period 0 and finances it through debt or taxes on agents' endowments. Debt is non-contingent, zero-coupon, and can have a one- or two-period maturity. We assume that even if the government does not issue two-period bonds, these exist in zero net supply. Thus, any effects of maturity structure are not driven by the government changing the set of tradable securities. We assume that all uncertainty is resolved in period 1 and there are only two states of nature, so markets are complete from the perspective of agents 1 and 2.

In Section 3 we consider the benchmark case where all agents can trade in period 0. Markets are then complete and the equilibrium is Pareto optimal. Moreover, a Ricardian equivalence result holds: consumption allocations and bond prices do not depend on the level or maturity structure of government debt. Suppose, for example, that the government lengthens the maturity structure by issuing fewer one-period and more two-period bonds. This shifts taxes in periods 1 and 2 towards the state where interest rates are low. The shift affects only agent 3, who is the only one to receive an endowment in periods 1 and 2, but the agent achieves the same consumption allocation by absorbing the incremental bond issuance.

In Section 4 we consider the more realistic case where the unborn agent 3 cannot trade in period 0. Markets are then incomplete and the maturity structure of government debt affects prices. The intuition is that any incremental bond issuance must be absorbed by agents 1 and 2. These agents, however, do not experience offsetting changes in future tax rates and therefore their consumption allocation is sensitive to changes in the maturity structure. For example, lengthening the maturity structure shifts consumption of agents 1 and 2 towards the state where interest rates are low. This lowers the valuation of both agents for two-period bonds, and raises the two-period interest rate and the slope of the yield curve.
We next determine the welfare-maximizing maturity structure. When agent 3 cannot trade in period 0, the mix of one- and two-period debt affects risksharing because it affects consumption allocations. For example, lengthening the maturity structure redistributes consumption towards agents 1 and 2, and away from agent 3 (through higher taxes), in the state when interest rates are low. We show that an optimal maturity structure implements the complete-markets allocation of Section 3. Thus, the government should issue the quantity of two-period bonds that agent 3 would sell if allowed to trade in period 0 in an economy where two-period bonds are in zero net supply. In other words, the government’s optimal issuance policy replicates the actions of private agents not present in the market.

The optimal maturity structure depends on agents’ preferences and endowments. Our main focus is to characterize how it depends on the clientele mix in period 0. Interpreting agent 1 as the short-horizon clientele and agent 2 as the long-horizon clientele, we characterize the mix by the fraction of endowment going to each agent. Intuition suggests that the optimal maturity structure should involve more short-term debt when the short-horizon clientele is wealthier. We confirm this intuition when agents’ coefficient of relative risk aversion $\gamma$ is larger than one, but find the opposite result when $\gamma < 1$ and no clientele effects when $\gamma = 1$ (log preferences). For example, when $\gamma = 1$, agents behave myopically and their portfolio choice is independent of the time when they need to consume. Asset-pricing research (e.g., equity premium puzzle) generally supports the assumption $\gamma > 1$. Since clientele effects seem prevalent in practice, our result generates a similar conclusion in the context of the term structure.\(^2\)

Our analysis relates to a long literature on optimal public debt policy. The benchmark in this literature is the Ricardian equivalence result of Barro (1974). Ricardian equivalence fails in the presence of distortionary taxation. Distortionary taxes imply an optimal time path for the level of government debt, as shown in Barro (1979) and Aiyagari, Marcet, Sargent and Seppäälä (2002). Distortionary taxes also imply an optimal composition of the government debt portfolio. Lucas and Stokey (1983) derive the optimal portfolio in terms of Arrow-Debreu securities. Angeletos (2002) and Buera and Nicolini (2004) show how the first-best outcome can be implemented with non-contingent bonds of different maturities, provided that there are bonds of as many maturities as states of nature so that markets are complete. Nosbusch (2008) derives the optimal maturity structure under incomplete markets. Faraglia, Marcet and Scott (2006) extend the complete markets analysis of optimal maturity structure to a framework with capital accumulation. The general idea in these papers is that the optimal debt portfolio is chosen so that its value is negatively correlated with shocks to government spending\(^3\). This allows the government to achieve tax smoothing

\(^{2}\)This assumes that clienteles are modeled through standard preferences, i.e., ignoring constraints or other institutional frictions.

\(^{3}\)A separate strand of the literature considers optimal debt policy when debt contracts are nominal and the
across states and over time, which is welfare improving when taxes are distortionary. Our analysis differs because taxes are non-distortionary and the role of maturity structure is to share risks across generations.\(^4\)

It is well known that Ricardian equivalence fails in models with overlapping generations (OLG), along the lines of Samuelson (1958), Diamond (1965) and Blanchard (1985). With overlapping generations, the timing of debt and taxes matters because debt shifts the tax burden to future generations.\(^5\) This holds even when generations are infinitely lived, as shown by Buiter (1988) and Weil (1989). Fischer (1983) and Gale (1990) show how debt can be used for risk-sharing across generations in a stochastic 2-period OLG model.\(^6\) Our analysis differs because we allow for heterogeneous investment horizons and clienteles.

The role of clienteles is emphasized in early term-structure hypotheses. The market-segmentation hypothesis of Culbertson (1957) and others posits that each maturity has its own clientele and constitutes a segmented market. The preferred-habitat hypothesis of Modigliani and Sutch (1966) posits that clienteles can engage in limited substitution across maturities. Vayanos and Vila (2007) develop a formal model of preferred habitat in which each maturity has its own clientele, and substitution across maturities is carried out by risk-averse arbitrageurs. They assume that clienteles are infinitely risk-averse over consumption at their desired maturity. We instead model clienteles through CRRA preferences, dispense with arbitrageurs, and perform a normative analysis of maturity structure.

Finally, our emphasis on clienteles in the debt market is consistent with recent empirical findings. Greenwood and Vayanos (2007) show that the average maturity of government debt is positively related to the slope of the yield curve, consistent with our theoretical results.\(^7\) Krishnamurthy and Vissing-Jorgensen (2007) show that the supply of government debt is negatively related to the corporate spread, i.e., government debt is expensive relative to corporate when it is in short supply. This is consistent with the existence of a clientele for government bonds.

government has control over inflation. In this case, the government can use state contingent inflation to achieve tax smoothing. Prominent examples of this approach include Lucas and Stokey (1983), Bohn (1988), Calvo and Guidotti (1990,1992), Barro (2002), Benigno and Woodford (2003), and Lustig, Sleet and Yeltekin (2006). Missale and Blanchard (1994) show that with nominal debt contracts, the optimal maturity can be decreasing in the total size of the debt.

\(^4\) An alternative way to achieve intergenerational risk-sharing is through the social security system. Ball and Mankiw (2007) show how social security can be used to implement risk-sharing contracts that all generations would be willing to sign if they were able to meet behind a “Rawlsian veil of ignorance.” Campbell and Nosbusch (2007) study how a social security system that shares risks across generations impacts equilibrium asset prices.

\(^5\) In the presence of capital, debt also matters because it affects capital accumulation.

\(^6\) Weiss (1980) and Bhattacharya (1982) show that in a stochastic OLG framework with money, state-contingent inflation can be used to share risks across generations.

\(^7\) This positive relationship would also arise in 2-period OLG models.
2 Model

There are three periods $t = 0, 1, 2$ and three agents $i = 1, 2, 3$. Figure 1 describes agents’ life-cycles, preferences and endowments.

Agent 1 is born in period 0 and lives for one period. Agent 2 is also born in period 0 but lives for two periods. Agent 3 is born in period 1 and lives for one period. Agents consume only in the last period of their lives, which is period 1 for agent 1 and period 2 for agents 2 and 3. Utility over consumption is CRRA with the same coefficient of relative risk aversion $\gamma$ for all agents. Agent $i$’s utility thus is

$$u(c_i) \equiv \frac{(c_i)^{1-\gamma}}{1-\gamma},$$

where $c_i$ is the agent’s consumption. The aggregate endowment in period $t$ is $e_t$. Agent 3 receives the entire endowment in periods 1 and 2. The endowment in period 0 is shared between agents 1 and 2, with agent 1 receiving a fraction $\alpha$.

Our assumptions on agents’ life-cycles, preferences and endowments can be motivated in reference to an infinite-horizon overlapping-generations setting. Suppose that agents live for three periods and consume in the last period of their lives. Then, in any given period three generations are alive: the young, the middle-aged, and the old. We can interpret agent 1 as the middle-aged generation in period 0, agent 2 as the young generation in period 0, and agent 3 as an aggregate of all future generations. Of course, considering a full-fledged overlapping-generations model rather than a three-period version, could be an interesting extension of our analysis.

Agents can invest in a one-period linear production technology with riskless return. The return between periods 0 and 1 is $r_{01}$, and that between periods 1 and 2 is $r_{12}$. The return $r_{12}$ becomes
known in period 1, and constitutes the only uncertainty in the model. We assume that \( r_{12} \) can take two values, \( r_{12} \) and \( \overline{r}_{12} \), with respective probabilities \( p \) and \( 1 - p \).

The government incurs an expenditure \( g_0 \) in period 0, financed through taxes or debt. Taxes are on agents’ endowments and are proportional. Debt can have a one- or two-period maturity. We denote by \( \tau_t \) the tax rate in period \( t \), by \( S_{01} \) and \( S_{02} \), respectively, the face value of one- and two-period bonds issued by the government in period 0, and by \( S_{12} \) the face value of one-period bonds issued by the government in period 1. The tax rates \( \tau_1 \) and \( \tau_2 \), as well as the face value \( S_{12} \) can depend on \( r_{12} \). From now on, we refer to the face values \( (S_{01}, S_{02}, S_{12}) \) as the bonds’ supplies.

Since \( r_{12} \) can take two values, two-period bonds complete the market from the viewpoint of agents trading in period 0. Thus, in issuing two-period bonds, the government can act as a financial innovator. Such innovation, however, can also be done by the private sector through, e.g., swaps or other interest-rate derivatives. To allow for innovation by the private sector, we assume that even if the government does not issue two-period bonds, this asset exists in zero net supply. Thus, any effects of maturity structure in our model are not driven by the government changing the set of tradable securities.

We assume that agents cannot run the production technology in reverse, i.e., investment in the technology must be non-negative. This imposes feasibility restrictions on the government’s expenditure and tax rates. The aggregate investment in the production technology in period 0 is \( e_0 - g_0 \) and is positive if

\[
e_0 > g_0. \tag{1}
\]

The aggregate investment in period 1 is \((e_0 - g_0)(1 + r_{01}) + e_1 - c_1 \). Since agent 1 can guarantee the consumption level \( \hat{c}_1 \equiv \alpha e_0 (1 - \tau_0)(1 + r_{01}) \) by investing in the production technology, \( c_1 \) cannot be smaller than \( \hat{c}_1 \) in both states. Therefore, a necessary condition for aggregate investment to be positive in period 1 is

\[
(e_0 - g_0)(1 + r_{01}) + e_1 > \alpha e_0 (1 - \tau_0)(1 + r_{01}). \tag{2}
\]

We assume (1) and (2) from now on.

Because agents can short one-period bonds, absence of arbitrage implies that these bonds must return at least as much as the production technology. Because, however, agents cannot run the technology in reverse, bonds can return more than the technology. If bonds do return more, then agents’ investment in the technology is zero. Eqs. (1) and (2) imply that zero investment is possible only in period 1 and at most one state. From now on, we define and compute equilibria under the
assumption that one-period bonds return the same as the technology. This assumption holds as long as exogenous parameters are restricted so that the aggregate investment in the technology is positive in both states in period 1.

3 Complete Markets

In this section we consider the benchmark case where agent 3 can trade in period 0. In this case markets are complete from the viewpoint of all agents and the equilibrium is Pareto optimal. Moreover, the maturity structure of government debt is irrelevant, not affecting prices and consumption allocations.

3.1 Definition of Equilibrium

We first define equilibrium and show the irrelevance of maturity structure. We denote by \( r_{02} \) the two-period interest rate, by \( x_i^t \) the investment of agent \( i \) in the technology in period \( t = 0, 1 \), by \( \theta_{01}^i \) and \( \theta_{02}^i \), respectively, the face value of one- and two-period bonds held by agent \( i \) in period 0, and by \( \theta_{12}^i \) the face value of new one-period bonds purchased by agent \( i \) in period 1.

The budget constraints of agent 1 in periods 0 and 1 are respectively

\[
\alpha e_0 (1 - \tau_0) = \theta_{01}^1 \left( 1 + \frac{\theta_{02}^1}{1 + r_{02}} \right)^2 + x_0^1,
\]

\[
c^1 = \theta_{01}^1 + \frac{\theta_{02}^1}{1 + r_{12}} + x_0^1 (1 + r_{01}).
\]

Combining these equations, we find agent 1’s intertemporal budget constraint:

\[
c^1 = \alpha e_0 (1 - \tau_0) (1 + r_{01}) + \theta_{12}^1 \left[ \frac{1}{1 + r_{12}} - \frac{1 + r_{01}}{(1 + r_{02})^2} \right]. \tag{3}
\]

The first term is the consumption that agent 1 can achieve by investing only in one-period bonds, and the second term corresponds to the excess return of two-period bonds relative to one-period
bonds. The budget constraints of agent 2 in periods 0, 1 and 2 are respectively

\[(1 - \alpha)e_0(1 - \tau_0) = \frac{\theta_{01}^2}{1 + r_{01}} + \frac{\theta_{02}^2}{(1 + r_{02})^2} + x_0^2,\]

\[\theta_{01}^2 + x_0^2(1 + r_{01}) = \frac{\theta_{12}^2}{1 + r_{12}} + x_1^2,\]

\[c^2 = \theta_{02}^2 + \theta_{12}^2 + x_1^2(1 + r_{12}).\]

Combining these equations, we find agent 2’s intertemporal budget constraint:

\[c^2 = (1 - \alpha)e_0(1 - \tau_0)(1 + r_{01})(1 + r_{12}) + \theta_{02}^2 \left[ 1 - \frac{(1 + r_{01})(1 + r_{12})}{(1 + r_{02})^2} \right]. \quad (4)\]

The first term is the consumption that agent 2 can achieve by investing in one-period bonds and rolling over, and the second term corresponds to the excess return of two-period bonds relative to one-period bonds. Similar calculations for agent 3 yield

\[c^3 = e_1(1 - \tau_1)(1 + r_{12}) + e_2(1 - \tau_2) + \theta_{02}^3 \left[ 1 - \frac{(1 + r_{01})(1 + r_{12})}{(1 + r_{02})^2} \right]. \quad (5)\]

Since one-period bonds return the same as the production technology, agents’ only decision variable is the investment \(\theta_{02}\) in two-period bonds. Agent \(i\) chooses \(\theta_{02}\) to maximize \(E u(c^i)\) subject to the intertemporal budget constraint (3) for \(i = 1\), (4) for \(i = 2\), and (5) for \(i = 3\). We denote this problem by \(P^i\).

The budget constraints of the government in periods 0, 1 and 2 are respectively

\[g_0 = e_0\tau_0 + \frac{S_{01}}{1 + r_{01}} + \frac{S_{02}}{(1 + r_{02})^2}, \quad (6)\]

\[S_{01} = e_1\tau_1 + \frac{S_{12}}{1 + r_{12}},\]

\[S_{02} + S_{12} = e_2\tau_2.\]

Combining these equations yields the government’s intertemporal budget constraint

\[g_0 = e_0\tau_0 + \frac{e_1\tau_1}{1 + r_{01}} + \frac{e_2\tau_2}{(1 + r_{01})(1 + r_{12})} + S_{02} \left[ \frac{1}{(1 + r_{02})^2} - \frac{1}{(1 + r_{01})(1 + r_{12})} \right]. \quad (7)\]
Definition 1. A complete-markets equilibrium consists of consumption allocations \( \{c^i\}_{i=1,2,3} \), holdings \( \{\theta^i_{02}\}_{i=1,2,3} \) in two-period bonds, and a two-period interest rate \( r_{02} \), such that

- \( \{c^i\}_{i=1,2,3} \) are given by the intertemporal budget constraints (3), (4), (5).
- \( \{\theta^i_{02}\}_{i=1,2,3} \) solve \( \mathcal{P}^i \).
- The government meets its intertemporal budget constraint (7).
- The market for two-period bonds clears:

\[
\sum_{i=1}^{3} \theta^i_{02} = S_{02}. \tag{8}
\]

3.2 Irrelevance Result

Suppose next that the government lengthens the maturity structure in period 0, raising the supply of two-period bonds to \( \hat{S}_{02} > S_{02} \). Suppose also that the tax rate \( \tau_0 \) is kept constant, which implies from (6) that the total market value of debt does not change. To show that the change in maturity structure does not affect prices, we show that the two-period interest rate \( r_{02} \) that clears the market under \( S_{02} \) does so under \( \hat{S}_{02} \) as well. The change in maturity structure impacts the tax rates \( \tau_1 \) and \( \tau_2 \). Indeed, since the government relies more on long-term debt and less on refinancing short-term debt, taxes shift towards the state where the interest rate \( r_{12} \) is low. The change in taxes does not affect agents 1 and 2 because they do not receive endowments in periods 1 and 2. Thus, if the two-period interest rate \( r_{02} \) does not change, agents 1 and 2 choose the same holdings \( \hat{\theta}^i_{02} \) of two-period bonds under \( \hat{S}_{02} \) as their holdings \( \theta^i_{02} \) under \( S_{02} \). On the other hand, agent 3 is affected by the change in taxes. To determine the effect, we eliminate \( \tau_1 \) and \( \tau_2 \) in agent 3’s budget constraint (5) by using the budget constraint (7) of the government:

\[
c^3 = e_1(1 + r_{12}) + e_2 - (g_0 - e_0 \tau_0)(1 + r_{01})(1 + r_{12}) + (\theta^3_{02} - S_{02}) \left[ 1 - \frac{(1 + r_{01})(1 + r_{12})}{(1 + r_{02})^2} \right]. \tag{9}
\]

Eq. (9) confirms that lengthening the maturity structure (i.e., raising \( S_{02} \)) makes agent 3 worse off in the state where \( r_{12} \) is low because of the change in taxes. Eq. (9) also shows that if \( r_{02} \) does not change, agent 3 chooses holdings \( \hat{\theta}^3_{02} \) in two-period bonds such that

\[
\hat{\theta}^3_{02} - \hat{S}_{02} = \theta^3_{02} - S_{02}. \tag{10}
\]
Thus, agent 3 absorbs the incremental bond issuance $\hat{S}_{02} - S_{02}$, and so offsets the change in taxes. The two-period interest rate $r_{02}$ clears the market under $\hat{S}_{02}$ because

$$\sum_{i=1}^{3} \hat{\theta}_{i02} = \sum_{i=1}^{3} \theta_{i02} + \hat{S}_{02} - S_{02} = \hat{S}_{02},$$

where the second step follows from (10) and the third from (8). Since the interest rate remains the same under $\hat{S}_{02}$, so do consumption allocations. This proves the following lemma:

**Lemma 1.** Suppose that agent 3 can trade in period 0. Then, given an equilibrium under $S_{02}$, there exists an equilibrium under $\hat{S}_{02} \neq S_{02}$ with identical prices and consumption allocations.

### 3.3 Prices and Allocations

We next solve for equilibrium prices and allocations. Since markets are complete, equilibrium is Pareto optimal. Therefore, agents’ consumption allocations $(c^1, c^2, c^3)$ maximize a weighted sum of utilities

$$E [\lambda u(c^1) + \mu u(c^2) + u(c^3)]$$

subject to the aggregate budget constraint

$$c^1(1 + r_{12}) + c^2 + c^3 = (e_0 - g_0)(1 + r_{01})(1 + r_{12}) + e_1(1 + r_{12}) + e_2.$$  

(Eq. (12) can be derived by adding the budget constraints (3), (4) and (9) of agents 1, 2 and 3.) The solution to this maximization problem, $S$, determines $(c^1, c^2, c^3)$ as function of $(\lambda, \mu)$. The weights $(\lambda, \mu)$ supporting the complete markets equilibrium allocation depend on agents’ endowments, and can be determined through agents’ budget constraints.

**Proposition 1.** Agents’ equilibrium consumption allocations are given by

$$c^1 = \frac{(e_0 - g_0)(1 + r_{01}) + e_1 + \frac{e^2}{1 + r_{12}}}{1 + \frac{1}{\lambda^\frac{1}{2}}(1 + r_{12})^{\frac{1}{2} - 1}},$$

$$c^2 = \frac{\mu^\frac{1}{2}}{\mu^\frac{1}{2} + 1} \frac{(e_0 - g_0)(1 + r_{01})(1 + r_{12}) + e_1(1 + r_{12}) + e_2}{1 + \lambda^\frac{1}{2}(1 + r_{12})^{1 - \frac{1}{2}}}$$

$$c^3 = \frac{1}{\mu^\frac{1}{2} + 1} \frac{(e_0 - g_0)(1 + r_{01})(1 + r_{12}) + e_1(1 + r_{12}) + e_2}{1 + \frac{1}{\lambda^\frac{1}{2}}(1 + r_{12})^{1 - \frac{1}{2}}}.$$
where \( \hat{\lambda} \equiv \lambda/(1 + \mu^{\frac{1}{\gamma}}) \). The weight \( \hat{\lambda} \) solves the non-linear equation
\[
\mathbb{E}[u'(c^1)c^1] = \alpha e_0(1 - \tau_0)(1 + r_{01})\mathbb{E}[u'(c^1)]
\] (16)
and the weight \( \mu \) is given by
\[
\mu = \left[ \frac{(1 - \alpha)e_0(1 - \tau_0)}{\frac{e_1}{1 + r_{01}} + \frac{e_2}{(1 + r_{02})^2} - (g_0 - e_0\tau_0)} \right]^{\gamma}.
\] (17)

The maximization problem \( S \) can be given an intuitive interpretation as a two-stage problem. In a first stage we take aggregate consumption in period 2 as given, and allocate it optimally between agents 2 and 3. In a second stage we allocate consumption between periods 1 and 2, i.e., between agent 1 and the aggregate of agents 2 and 3. Because of CRRA utility, the objective function in the second stage takes the form
\[
\mathbb{E}[\hat{\lambda}u(c^1) + u(c^2 + c^3)]
\] (18)
for the weight \( \hat{\lambda} \) defined in Proposition 1. Eq. (18) is the objective function of a representative agent trading off consumption between periods 1 and 2 under CRRA utility. The representative agent’s discount factor is \( 1/\hat{\lambda} \), the inverse of the Pareto weight of agent 1 relative to the aggregate of agents 2 and 3.

Lemma 2 compares agents’ consumption levels under high and low interest rates. This comparison helps determine the relative returns of one- and two-period bonds, as well as agents’ holdings of two-period bonds.

**Lemma 2.** Agents 2 and 3 consume more in state \( r_{12} \) than in state \( r_{12} \). There exists \( \hat{\gamma} > 1 \) such that agent 1 consumes more in state \( r_{12} \) than in state \( r_{12} \) if \( \gamma > \hat{\gamma} \), but the comparison is reversed if \( \gamma < \hat{\gamma} \).

The effect of interest rates on consumption can be determined from the maximization of the representative agent’s utility function (18). Because high interest rates make the representative agent better off, they raise consumption in period 2, i.e., for agents 2 and 3. The effect on consumption in period 1 is ambiguous. High interest rates induce the representative agent to substitute consumption towards period 2, and this tends to reduce consumption in period 1. At the same time, the income effect induces the representative agent to raise consumption in both periods. Finally, the income effect is tempered by a wealth effect arising because high interest rates reduce the
present value of the representative agent’s endowment, evaluated as of period 1. When \( \gamma \) is larger than some value \( \hat{\gamma} \) (with \( \hat{\gamma} > 1 \)), the income effect dominates, and therefore agent 1 consumes more under high interest rates. When instead \( \gamma < \hat{\gamma} \), the substitution and wealth effects dominate.

The relative returns of one- and two-period bonds can be determined from agents’ consumption. The first-order condition of agents 2 and 3 is

\[
\mathbb{E} \left\{ u'(c^i) \left[ (1 + r_{02})^2 - (1 + r_{01})(1 + r_{12}) \right] \right\} = 0
\quad (19)
\]

for \( i = 2, 3 \). The term in square brackets is the excess return of two-period bonds relative to the strategy of investing in one-period bonds and rolling over. This return is high when \( r_{12} \) is low, which is also when the consumption of agents 2 and 3 is low. Therefore, two-period bonds are a valuable hedge from agent 2 and 3’s viewpoint, and return less on average than one-period bonds over a two-period horizon:

\[
(1 + r_{02})^2 < \mathbb{E} [(1 + r_{01})(1 + r_{12})]. \quad (20)
\]

To compare returns over a one-period horizon, we use the first-order condition of agent 1, which is

\[
\mathbb{E} \left\{ u'(c^1) \left[ \frac{(1 + r_{02})^2}{1 + r_{12}} - (1 + r_{01}) \right] \right\} = 0.
\quad (21)
\]

When \( \gamma \) is large, agent 1 consumes less when \( r_{12} \) is low. Therefore, two-period bonds are a valuable hedge from the viewpoint of that agent as well, and return less on average than one-period bonds over a one-period horizon:\(^8\)

\[
\mathbb{E} \left[ \frac{(1 + r_{02})^2}{1 + r_{12}} \right] < 1 + r_{01}. \quad (22)
\]

This inequality is, however, reversed when \( \gamma \leq \hat{\gamma} \). Thus, it is possible that two-period bonds outperform on average one-period bonds over a one-period horizon, while underperforming over two periods.\(^9\)

Agents’ holdings of two-period bonds can be determined from their consumption. For example, when \( \gamma \) is large, agent 1 consumes more when interest rates are high, and is thus a seller of two-period bonds.\(^8\)

---

\(^8\)This argument applies in equilibrium because agent 1’s consumption under autarchy is riskless. In equilibrium agent 1 prefers to take interest-rate risk through a short position in two-period bonds, because this generates a positive excess return.

\(^9\)Mathematically, (20) can be consistent with the reverse of (22) because of Jensen’s inequality.
4 Incomplete Markets

We next consider the case where agent 3 cannot trade in period 0. In this case markets are incomplete, and the maturity structure of government debt affects prices and allocations.

4.1 Definition of Equilibrium

When agent 3 cannot trade in period 0, he invests his period 1 endowment in one-period bonds, and thus performs no optimization over portfolios. The agent’s intertemporal budget constraint is

\[ c^3 = e_1(1 - \tau_1)(1 + r_{12}) + e_2(1 - \tau_2), \]  

(23)

and is derived from its complete-markets counterpart (5) by setting \( \theta^3_{02} = 0 \). The market-clearing equation for two-period bonds is derived from (8) in the same manner.

**Definition 2.** An incomplete-markets equilibrium consists of consumption allocations \( \{c^i\}_{i=1,2,3} \), holdings \( \{\theta^i_{02}\}_{i=1,2} \) in two-period bonds by agents 1 and 2, and a two-period interest rate \( r_{02} \), such that

- \( \{c^i\}_{i=1,2,3} \) are given by the intertemporal budget constraints (3), (4), (23).
- \( \{\theta^i_{02}\}_{i=1,2} \) solve \( \mathcal{P}^i \).
- The government meets its intertemporal budget constraint (7).
- The market for two-period bonds clears:

\[ \sum_{i=1}^{2} \theta^i_{02} = S_{02}. \]  

(24)

4.2 Effects of Maturity Structure

Suppose next that the government raises the supply \( S_{02} \) of two-period bonds, while keeping the total market value of debt constant. The new issuance must be absorbed by agents 1 and 2, but these agents are not affected by the tax changes caused by the issuance. Therefore, issuance has no effect on their demand for two-period bonds, unless \( r_{02} \) changes. To determine the impact of issuance on \( r_{02} \), we next solve for equilibrium prices and allocations.
Substituting agent 1’s intertemporal budget constraint (3) into his first-order condition (21), we find

\[ E \left[ u' \left( A_1 + \theta_{02}^1 (y - \omega) \right) (y - \omega) \right] = 0, \quad (25) \]

where

\[ A_1 \equiv \alpha e_0 (1 - \tau_0) (1 + r_{01}), \]
\[ y \equiv \frac{1}{1 + r_{12}}, \]
\[ \omega \equiv \frac{1 + r_{01}}{(1 + r_{02})^2}. \]

Eq. (25) links agent 1’s demand for two-period bonds to the two-period interest rate. Since \( u'(0) = \infty \), (25) has a solution \( \theta_{02}^1 \) for any value of \( r_{02} \) satisfying no-arbitrage. Moreover, the solution is unique since utility is strictly concave. Agent 2’s demand for two-period bonds, \( \theta_{02}^2 \), is similarly the unique solution of

\[ E \left[ u' \left( A_2 z + \theta_{02}^2 (1 - z \omega) \right) (1 - z \omega) \right] = 0, \quad (26) \]

where

\[ A_2 \equiv (1 - \alpha) e_0 (1 - \tau_0) (1 + r_{01}), \]
\[ z \equiv (1 + r_{12}). \]

Following standard arguments in portfolio theory, we can show that demands are increasing in \( r_{02} \) and converge to \(-\infty\) and \( \infty \) when \( r_{02} \) reaches its arbitrage bounds.

**Lemma 3.** The demands \( \theta_{02}^1 \) and \( \theta_{02}^2 \) are increasing in \( r_{02} \), converge to \(-\infty\) when \((1 + r_{02})^2\) goes to \((1 + r_{01})(1 + r_{12})\), and converge to \( \infty \) when \((1 + r_{02})^2\) goes to \((1 + r_{01})(1 + r_{12})\).

Lemma 3 implies that there exists a unique value of \( r_{02} \) that equates the demand for two-period bonds to their supply \( S_{02} \). Moreover, this value is increasing in supply because demand is increasing in \( r_{02} \). Thus, an increase in supply raises the yield of two-period bonds and the slope of the term structure.

**Proposition 2.** There exists a unique equilibrium. An increase in the supply \( S_{02} \) of two-period bonds raises the equilibrium two-period interest rate \( r_{02} \).
Since an increase in supply raises \( r_{02} \), it also raises the expected excess return of two-period bonds relative to one-period bonds. Recall that under complete markets, this return is negative over a two-period horizon, and possibly over a one-period horizon as well. Under incomplete markets, this return can be positive or negative, depending on \( S_{02} \). If two-period bonds are in large supply, then they trade at a low price in period 0, and their expected return exceeds that of one-period bonds over a one- and a two-period horizon. Conversely, if supply is small, then two-period bonds return less on average than one-period bonds.

While excess returns under incomplete markets are typically different than under complete markets, there exists a value of \( S_{02} \) for which they are the same. Indeed, suppose that the government sets \( S_{02} \) equal to the quantity \( S^*_0 \) of two-period bonds that agents 1 and 2 buy in the complete-markets equilibrium. Then, \( r_{02} \) is as in that equilibrium, and so are excess returns. The observation that the government can replicate the complete-markets allocation plays a crucial role in our analysis of optimal maturity structure.

### 4.3 Optimal Maturity Structure

When markets are incomplete, changes in maturity structure affect the allocation of risk among agents. Suppose, for example, that the government increases the supply of two-period bonds, keeping the total market value of debt constant. Since agents 1 and 2 buy these bonds, their consumption increases in the state where interest rates are low. The increase in consumption is financed from the bond payouts, which, in turn, are financed from taxes in periods 1 and 2 paid by agent 3. This redistributes consumption towards agents 1 and 2, and away from agent 3, in the state where interest rates are low, and the converse redistribution occurs in the state where rates are high.

Since changes in maturity structure affect risk-sharing, they also affect welfare. We next determine welfare-maximizing maturity structures and their properties. We allow the government to optimize over the relative supplies of one- and two-period bonds in period 0, keeping constant the total market value of debt. Thus, choosing a maturity structure amounts to fixing the tax rate \( \tau_0 \) and optimizing over \( S_{02} \).

We denote the set of equilibrium allocations generated by different values of \( S_{02} \) by \( \mathcal{A} \), and define Pareto optimal maturity structures as those corresponding to the Pareto frontier of \( \mathcal{A} \). A welfare-maximizing maturity structure must be Pareto optimal, otherwise the government could achieve a Pareto improvement by choosing a different value of \( S_{02} \).

The set of Pareto optimal maturity structures includes \( S^*_0 \), the quantity of two-period bonds that agents 1 and 2 buy in the complete-markets equilibrium. Indeed, the allocation under \( S^*_0 \)
coincides with the complete-markets allocation. Therefore, it belongs not only to the Pareto frontier of $A$, but also to the frontier of the larger set $A_0$ of all feasible allocations.\footnote{The set $A_0$ is larger than $A$ because it includes allocations that can be achieved through general redistributions among agents. Allocations in $A$, by contrast, can be achieved only through the redistributions implicit in changing $S_{02}$.}

In addition to $S_{02}$, there may exist other Pareto optimal maturity structures. To select among them, we refine our optimality criterion, ruling out not only Pareto improvements but also aggregate gains. That is, we define a maturity structure to be optimal if changes in $S_{02}$ cannot benefit winners more than they hurt losers. To ensure that gains and losses are comparable across agents, we measure them in monetary terms as of period 0. More precisely, consider a change from $S_{02}$ to $\hat{S}_{02}$, and denote by $c^i(S_{02})$ and $c^i(\hat{S}_{02})$ the consumption of agent $i$ in the respective equilibria. Define the gain $T^i$ of agent $i$ as the investment in one-period bonds that the agent would forego in period 0 to remain with the same utility as before the change.\footnote{The gain $T^i$ is analogous to the concept of compensating variation (e.g., Varian (1992)). Note that while agent 3 is not trading in period 0, we can interpret $T^3$ as the present value of one-period bonds that the agent would forego in period 1.} Thus, the gain of agent 1 is given by

$$
 Eu\left[c^1(S_{02})\right] \equiv Eu\left[c^1(\hat{S}_{02}) - (1 + r_{01})T^1\right], \quad (27)
$$

and that of agent $i = 2, 3$ by

$$
 Eu\left[c^i(S_{02})\right] \equiv Eu\left[c^i(\hat{S}_{02}) - (1 + r_{01})(1 + r_{12})T^i\right]. \quad (28)
$$

**Definition 3.** A maturity structure $S_{02}$ is optimal if there does not exist $\hat{S}_{02}$ such that $\sum_{i=1}^{3} T^i > 0$.

The maturity structure $S^*_{02}$ satisfies Definition 3. Indeed, if it is not optimal, then an aggregate gain can be achieved through an alternative choice $\hat{S}_{02}$. Using appropriate transfers from winners to losers, we can modify the equilibrium under $\hat{S}_{02}$ to construct an allocation that Pareto dominates that under $S^*_{02}$. This is a contradiction because the latter allocation is in the Pareto frontier of $A_0$, i.e., is Pareto optimal among all feasible allocations. Proposition 3 shows that $S^*_{02}$ is the unique optimal maturity structure. This means that allocations under other Pareto optimal maturity structures are not in the Pareto frontier of $A_0$ (while being in that of $A$). Under such maturity structures, the non-participation of agent 3 in period 0 impairs risksharing. Changing the maturity structure to $S^*_{02}$ renders non-participation inconsequential, and achieves aggregate gains.

**Proposition 3.** The unique optimal maturity structure is $S^*_{02}$.
markets equilibrium, it is also the quantity sold by agent 3 and the government. This suggests
the following intuitive interpretation of optimal maturity structure. Suppose that in the complete-
markets equilibrium the government is absent from the market for two-period bonds, using only
one-period financing. Agent 3 then sells two-period bonds in quantity $S_{02}^*$. When instead agent 3
cannot trade in period 0, the government can replicate the complete-markets allocation by issuing
two-period bonds in the same quantity $S_{02}^*$. Thus, the government can raise welfare through its
choice of maturity structure because it can replicate the actions of private agents not present in
the market. Through this replication, it renders non-participation inconsequential and eliminates
the market incompleteness.

We next derive properties of the optimal maturity structure by exploiting the link with the
complete-markets equilibrium. A first property concerns the excess returns of two-period bonds.

**Proposition 4.** Under the optimal maturity structure, two-period bonds earn lower returns on
average than one-period bonds over a two-period horizon. They also earn lower returns on average
over a one-period horizon if $\gamma$ is sufficiently large, but higher returns if $\gamma \leq \hat{\gamma}$.

Proposition 4 suggests that positive excess returns of long-term bonds can be a symptom of
non-optimal maturity structures, i.e., long-term bonds being in excessively large supply. Indeed,
current generations require positive excess returns from long-term bonds if they consume relatively
less when interest rates are high. At the same time, future generations consume more when interest
rates are high because they earn a high return on their endowments. Thus, the consumption of
current and future generations covaries negatively, implying inefficient risksharing. Risksharing
could be made efficient if markets were complete and future generations could trade today. But
alternatively, the government can improve risksharing by shortening the maturity structure. Indeed,
replacing long-term bonds by short-term bonds raises the consumption of current generations when
interest rates are high, while also reducing taxes of future generations when interest rates are low.

A related implication is that the government should not necessarily strive to equalize expected
returns across bonds of different maturities. Suppose, for example, that expected returns of long-
term bonds are below those for short-term bonds (as seems to be the case currently in the UK).
The government could respond by tilting issuance towards long-term bonds because this lowers the
expected return of its debt liabilities, thus lowering expected funding costs. Our model suggests,
however, that minimizing expected funding costs should not be the only objective. Indeed, long-
term bonds have an implicit cost: taxes for future generations must increase when interest rates
are low, which is also when consumption of future generations is low.

While Proposition 4 characterizes the excess returns of two-period bonds under the optimal
maturity structure, it does not determine the supply of these bonds. A partial characterization of
optimal supply is in Proposition 5.

**Proposition 5.** If $\gamma$ is large, then the optimal supply $S_{02}$ of two-period bonds is positive for small $\alpha$ and negative for large $\alpha$. If $\gamma \leq \hat{\gamma}$, then the optimal supply of two-period bonds is positive.

The intuition is as follows. If $\gamma$ is large, then agents 2 and 3 have a strong motive to hedge against low interest rates. Moreover, this motive is relatively stronger for agent 2, who consumes entirely out of savings, than for agent 3, who consumes partly out of current income. Therefore, when markets are complete, agent 3 insures agent 2 against low interest rates by selling him two-period bonds, while agent 1 sells two-period bonds to both agents 2 and 3. When agent 1 is relatively unimportant (small $\alpha$), the first effect dominates, and agent 3 is a net seller of two-period bonds. As a result, the government should be a net issuer of two-period bonds under incomplete markets. When instead agent 2 is relatively unimportant (large $\alpha$), the second effect dominates, and agent 3 is a net buyer of two-period bonds. As a result, the government should be a net investor in two-period bonds (and finance its investment through one-period debt). Finally, if $\gamma \leq \hat{\gamma}$, then agent 1 is a buyer of two-period bonds under complete markets. Since, in addition, agent 2 values two-period bonds more than agent 3, agent 3 is a net seller. As a result, the government should be a net issuer of two-period bonds.

### 4.4 Clientele Effects

Proposition 5 implies that the optimal maturity structure depends on the relative importance of agents 1 and 2, i.e., the short- and long-horizon clienteles. We next characterize clientele effects in more detail, describing the clientele mix by the fraction of period 0 endowment going to each agent, i.e., $\alpha$ to agent 1 and $1 - \alpha$ to agent 2.

**Proposition 6.** If $\gamma > 1$, then a decrease in $\alpha$ (i.e., increase in the long-horizon clientele)

- Lowers the two-period interest rate in the complete- and incomplete-markets equilibria.
- Raises the optimal supply $S_{02}^*$ of two-period bonds.

The results are reversed if $\gamma < 1$.

According to the preferred-habitat hypothesis (Modigliani and Sutch (1966)), short-term bonds are demanded mainly by short-horizon investors, i.e., agent 1 in our model, while long-term bonds are demanded by long-horizon investors, i.e., agent 2. Therefore, when agent 2 commands more resources, two-period bonds should be in higher demand and thus more expensive. Proposition 6 confirms this intuition when agents’ coefficient of relative risk aversion $\gamma$ is larger than one. When
\( \gamma < 1 \), however, the result is reversed, and when \( \gamma = 1 \) (logarithmic utility) the clientele mix has no effect. The assumption \( \gamma > 1 \) ensures that agent 2 invests a higher share of his endowment in two-period bonds than agent 1. When instead utility is logarithmic, agents behave myopically and their portfolio choice is independent of the time when they need to consume. Our results imply that utility functions with \( \gamma \leq 1 \) (and in particular, the logarithmic utility commonly used in term-structure models) do not generate preferred-habitat or clientele effects. Since such effects seem important in practice, our results support the assumption \( \gamma > 1 \).

Proposition 6 determines the impact of clienteles not only on bond prices, but also on the optimal maturity structure. As with prices, the results are consistent with practical intuition when \( \gamma > 1 \). Namely, when \( \gamma > 1 \), an increase in the clientele for two-period bonds raises the prices of these bonds and prompts the government to lengthen the maturity structure. The result is reversed when \( \gamma < 1 \), and the clientele mix has no effect on the optimal maturity structure when \( \gamma = 1 \).

While the results for \( \gamma > 1 \) and \( \gamma < 1 \) are opposites, they have a common implication: when changes to the clientele mix raise the price of two-period bonds, they also prompt the government to issue more such bonds. Thus, a welfare-maximizing government can appear to respond to prices in a way consistent with minimizing expected funding costs. As argued in Section 4.3, however, funding-cost minimization does not take into account the welfare of future generations.

## 5 Conclusion

This paper provides a novel theory of optimal maturity structure based on clienteles. Clienteles arise endogenously because generations differ in their consumption horizon. In this setting, the maturity structure of government debt affects welfare: the government can improve intergenerational risksharing by effectively replicating the actions of future generations, thereby alleviating a fundamental limited participation problem. Our setting is also appropriate for the analysis of demand and supply effects on the yield curve. For instance, our model provides a very natural explanation of why a lengthening of the maturity structure is typically accompanied by an increase in the slope of the curve.

We are currently working on establishing the robustness of our results by extending our framework along a certain number of dimensions. One extension is to allow endowments to be stochastic (or equivalently to introduce a risky asset in positive net supply into the economy). Another extension is to have agents consume in every period of their life. Finally, our analysis can be extended to an infinite horizon overlapping generations setting. In future work, we also intend to explore further the implications of our framework for the desirability of debt maturity policies that attempt to minimize expected interest costs.
APPENDIX

Proof of Proposition 1: The first-order condition of $S$ is

$$\frac{\lambda u'(c^1)}{1 + r_{12}} = \mu u'(c^2) = u'(c^3).$$  \hspace{1cm} (A.1)

Combining (12) with (A.1) and using the fact that utility is CRRA, we find (13)-(15). To derive (17), we multiply (4) and (9) by $u'(c^2)$, and take expectations over states. For (4), this yields

$$E\left[u'(c^2)c^2\right] = (1 - \alpha)e_0(1 - \tau_0)(1 + r_{01})E\left[u'(c^2)(1 + r_{12})\right] + \theta_{02}E\left[u'(c^2)\left[1 - \frac{(1 + r_{01})(1 + r_{12})}{(1 + r_{02})^2}\right]\right]$$

$$= (1 - \alpha)e_0(1 - \tau_0)(1 + r_{02})E\left[u'(c^2)\right],$$  \hspace{1cm} (A.2)

where the second step follows from (19). For (9), this yields

$$E\left[u'(c^2)c^3\right] = e_1\frac{(1 + r_{02})^2}{1 + r_{01}} + e_2 - (g_0 - e_0\tau_0)(1 + r_{02})^2E\left[u'(c^2)\right].$$  \hspace{1cm} (A.3)

Dividing (A.2) by (A.3), we find (17). To derive (16), we multiply (3) by $u'(c^1)$, and take expectations over states. This yields

$$E\left[u'(c^1)c^1\right] = \alpha e_0(1 - \tau_0)(1 + r_{01})E\left[u'(c^1)\right] + \theta_{01}E\left[u'(c^1)\left[\frac{1}{1 + r_{12}} - \frac{1 + r_{01}}{(1 + r_{02})^2}\right]\right]$$

$$= \alpha e_0(1 - \tau_0)(1 + r_{01})E\left[u'(c^1)\right],$$  \hspace{1cm} (A.4)

where the second step follows from (21). Eq. (A.4) coincides with (16).

Proof of Lemma 2: We denote by $\tau^i$ and $\xi^i$ respectively the consumption of agent $i$ in state $\tau_{12}$ and $\tau_{12}$. To prove the first statement, we proceed by contradiction. Suppose that $\tau^i \leq \xi^i$ for $i = 2$ or $i = 3$. Eq. (A.1) then implies that $\tau^i \leq \xi^i$ for $i = 1, 2, 3$. Subtracting (12) for state $\tau_{12}$ from the same equation for state $\tau_{12}$, we find

$$\tau^2 + \tau^3 - (\xi^2 + \xi^3) = [(e_0 - g_0)(1 + r_{01}) + e_1] (\tau_{12} - \tau_{12}) + \xi^1(1 + \tau_{12}) - \tau^1(1 + \tau_{12})$$

$$\geq [(e_0 - g_0)(1 + r_{01}) + e_1 - \tau^1] (\tau_{12} - \tau_{12}),$$  \hspace{1cm} (A.5)
where the second step follows from \( \bar{c}^1 \leq \epsilon^1 \). Eq. (A.4) implies that

\[
\bar{c}^1 \leq \alpha e_0 (1 - \tau_0) (1 + r_{01}) \leq \epsilon^1.
\]

Substituting into (A.5), we find

\[
\bar{c}^2 + \bar{c}^3 - (\epsilon^2 + \epsilon^3) \geq [(e_0 - g_0) (1 + r_{01}) + e_1 - \alpha e_0 (1 - \tau_0) (1 + r_{01})] (\bar{r}_{12} - \epsilon_{12}).
\] (A.6)

Eqs. (2) and (A.6) imply that \( \bar{c}^2 + \bar{c}^3 > \epsilon^2 + \epsilon^3 \), contradicting our original hypothesis. Therefore, \( \bar{c}^i > \epsilon^i \) for \( i = 2, 3 \).

Eq. (13) implies that \( \bar{c}^1 < \epsilon^1 \) if \( \gamma \leq 1 \). To show that \( \bar{c}^1 > \epsilon^1 \) if \( \gamma \) is large, we note that the argument establishing \( \bar{c}^2 + \bar{c}^3 > \epsilon^2 + \epsilon^3 \) applies also for \( \gamma = \infty \). Eq. (A.1) then implies that \( \bar{c}^1 > \epsilon^1 \) for \( \gamma = \infty \). Therefore, \( \bar{c}^1 > \epsilon^1 \) if \( \gamma \) is large.

**Proof of Lemma 3:** We first show that demands are increasing in \( r_{02} \). Differentiating (25) implicitly with respect to \( \omega \), we find

\[
\frac{\partial \theta_{02}^1}{\partial \omega} = \frac{\mathbb{E} u' [A_1 + \theta_{02}^1 (y - \omega)] + \mathbb{E} [u'' [A_1 + \theta_{02}^1 (y - \omega)] \theta_{02}^1 (y - \omega)]}{\mathbb{E} [u'' [A_1 + \theta_{02}^1 (y - \omega)] (y - \omega)^2]}.\]

Since \( u' > 0 \) and \( u'' < 0 \), the sign of \( \partial \theta_{02}^1 / \partial \omega \) is negative if

\[
\mathbb{E} [u'' [A_1 + \theta_{02}^1 (y - \omega)] \theta_{02}^1 (y - \omega)] > 0
\]

\[
\iff \mathbb{E} [F(y) u' [A_1 + \theta_{02}^1 (y - \omega)] (y - \omega)] > 0,\] (A.7)

where

\[
F(y) = \frac{u'' [A_1 + \theta_{02}^1 (y - \omega)]}{u' [A_1 + \theta_{02}^1 (y - \omega)]} \theta_{02}^1.
\]

Eq. (A.7) follows from (25) if the function \( F(y) \) is increasing. Because of CRRA utility,

\[
F'(y) = \frac{d}{dy} \left[ -\frac{\gamma \theta_{02}^1}{A_1 + \theta_{02}^1 (y - \omega)} \right] = \frac{\gamma (\theta_{02}^1)^2}{[A_1 + \theta_{02}^1 (y - \omega)]^2} > 0.
\]

Since the sign of \( \partial \theta_{02}^1 / \partial \omega \) is negative, that of \( \partial \theta_{02}^1 / \partial r_{02} \) is positive.
Differentiating (26) implicitly with respect to $\omega$, we find

$$\frac{\partial \theta_{02}^2}{\partial \omega} = \frac{\mathbb{E} \left[ u' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right] z \right] + \mathbb{E} \left[ u'' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right] \theta_{02}^2z(1 - z\omega) \right]}{\mathbb{E} \left[ u'' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right] (1 - z\omega)^2 \right]}.$$ 

Since $u' > 0$ and $u'' < 0$, the sign of $\partial \theta_{02}^2/\partial \omega$ is negative if

$$\mathbb{E} \left[ u'' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right] \theta_{02}^2z(1 - z\omega) \right] > 0 \iff \mathbb{E} \left[ G(z)u' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right] (1 - z\omega) \right] > 0,$$

(A.8)

where

$$G(z) \equiv \frac{u'' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right]}{u' \left[ A_2z + \theta_{02}^2(1 - z\omega) \right]} \theta_{02}^2z.$$

Eq. (A.8) follows from (26) if the function $G(z)$ is decreasing. Because of CRRA utility,

$$G'(z) = \frac{d}{dz} \left[ -\frac{\gamma \theta_{02}^2 z}{A_2z + \theta_{02}^2(1 - z\omega)} \right] = -\frac{\gamma \theta_{02}^2}{A_2z + \theta_{02}^2(1 - z\omega)} < 0.$$

Since the sign of $\partial \theta_{02}^2/\partial \omega$ is negative, that of $\partial \theta_{02}^2/\partial r_{02}$ is positive.

We next determine the asymptotic behavior of the demands when $r_{02}$ reaches its arbitrage bounds. Since demands are increasing, they converge to a limit when $(1+r_{02})^2 \rightarrow (1+r_{01})(1+r_{12})$.

To show that the limit is $\infty$, we proceed by contradiction. Suppose, for example, that $\theta_{02}^1$ has a finite limit. Denoting by $\overline{y}$ and $\underline{y}$ respectively the value of $y$ in state $\tau_{12}$ and $\underline{\tau}_{12}$, we can write (25) as

$$pu' \left[ A_1 + \theta_{02}^1(y - \omega) \right] (\overline{y} - \omega) + (1-p)u' \left[ A_1 + \theta_{02}^1(y - \omega) \right] (\underline{y} - \omega) = 0.$$  

(A.9)

The case $(1+r_{02})^2 \rightarrow (1+r_{01})(1+r_{12})$ corresponds to $\omega \rightarrow \overline{y}$. If $\theta_{02}^1$ has a finite limit $\overline{\theta}_{02}^1$, the limit of (A.9) is

$$(1-p)u' \left[ A_1 + \overline{\theta}_{02}^1(y - \overline{y}) \right] (\underline{y} - \overline{y}) = 0.$$  

Since the left-hand side is positive, we find a contradiction. Therefore, $\theta_{02}^1$ converges to $\infty$ and
so does $\theta_{02}^2$. A similar argument establishes that $\theta_{02}^1$ and $\theta_{02}^2$ converge to $-\infty$ when $(1 + r_{02})^2 \to (1 + r_{01})(1 + \Sigma_{12})$.

**Proof of Proposition 2:** Lemma 3 implies that the aggregate demand $\sum_{i=1}^{2} \theta_{02}^i(r_{02})$ is increasing in $r_{02}$, and its values range from $-\infty$ to $\infty$. Therefore, the market-clearing equation (24) has a unique solution $r_{02}$, which, in addition, is increasing in the supply $S_{02}$.

**Proof of Proposition 3:** We first show (formalizing the argument that precedes the statement of the proposition) that $S_{02}^*$ is optimal. If $S_{02}^*$ is not optimal, there exists $\hat{S}_{02}$ such that $\sum_{i=1}^{3} T_i > 0$.

The allocation

$$c^1 = c^1(\hat{S}_{02}) - (1 + r_{01})T^1,$$

$$c^2 = c^2(\hat{S}_{02}) - (1 + r_{01})(1 + r_{12})T^2,$$

$$c^3 = c^3(\hat{S}_{02}) + (1 + r_{01})(1 + r_{12})(T^1 + T^2),$$

satisfies the aggregate budget constraint (12). Moreover, agents 1 and 2 are equally well off as under $(c^1(S_{02}^*), c^2(S_{02}^*), c^3(S_{02}^*))$ because of (27) and (28), while agent 3 is better off because of (28) and $\sum_{i=1}^{3} T_i > 0$. This is a contradiction since $(c^1(S_{02}^*), c^2(S_{02}^*), c^3(S_{02}^*))$ maximizes (11) among all allocations satisfying (12).

Consider next a maturity structure $S_{02}$ and an infinitesimal change to $S_{02} + dS_{02}$. Eq. (27) implies that the infinitesimal gain $dT^1$ of agent 1 is given by

$$dT^1 = \frac{1}{\mathbb{E}u'[c^1(S_{02})]} \frac{d\mathbb{E}u[c^1(S_{02})]}{dS_{02}} dS_{02}. \tag{A.10}$$

Similarly, the infinitesimal gain $dT^i$ of agent $i = 2, 3$ is

$$dT^i = \frac{1}{\mathbb{E}u'[c^i(S_{02})]} \frac{d\mathbb{E}u[c^i(S_{02})]}{dS_{02}} dS_{02}. \tag{A.11}$$
Differentiating $E_u \left[ c^1(S_{02}) \right]$ with respect to $S_{02}$, we find

$$\frac{dE_u \left[ c^1(S_{02}) \right]}{dS_{02}} = E \left[ u' \left[ c^1(S_{02}) \right] \left[ -\theta_{02}^1 \frac{d\omega}{dS_{02}} + \frac{d\theta_{02}^1}{dS_{02}} (y - \omega) \right] \right]$$

$$= -E u' \left[ c^1(S_{02}) \right] \theta_{02}^1 \frac{d\omega}{dS_{02}},$$

where the first step follows from (3) and the second from (21). Substituting into (A.10), we find

$$dT^1 = -\frac{\theta_{02}^1}{1 + r_{01}} \frac{d\omega}{dS_{02}} dS_{02}. \quad (A.12)$$

Differentiating $E_u \left[ c^2(S_{02}) \right]$ with respect to $S_{02}$, we find

$$\frac{dE_u \left[ c^2(S_{02}) \right]}{dS_{02}} = E \left[ u' \left[ c^2(S_{02}) \right] \left[ -\theta_{02}^2 z \frac{d\omega}{dS_{02}} + \frac{d\theta_{02}^2}{dS_{02}} (1 - z\omega) \right] \right]$$

$$= -E u' \left[ c^2(S_{02}) \right] \theta_{02}^2 \frac{d\omega}{dS_{02}},$$

where the first step follows from (4) and the second from (19). Substituting into (A.11), we find

$$dT^2 = -\frac{\theta_{02}^2}{1 + r_{01}} \frac{d\omega}{dS_{02}} dS_{02}. \quad (A.13)$$

Differentiating $E_u \left[ c^3(S_{02}) \right]$ with respect to $S_{02}$, and using

$$c^3(S_{02}) = e_1(1 + r_{12}) + e_2 - (g_0 - e_0 \tau_0)(1 + r_{01})(1 + r_{12}) - S_{02} \left[ 1 - \frac{(1 + r_{01})(1 + r_{12})}{(1 + r_{02})^2} \right]$$

(which follows by setting $\theta_{02}^3 = 0$ in (9)), we find

$$\frac{dE_u \left[ c^3(S_{02}) \right]}{dS_{02}} = E \left[ u' \left[ c^3(S_{02}) \right] \left[ S_{02} z \frac{d\omega}{dS_{02}} - (1 - z\omega) \right] \right].$$

Substituting into (A.11), we find

$$dT^3 = \left[ \frac{S_{02}}{1 + r_{01}} \frac{d\omega}{dS_{02}} - \frac{E \left[ u' \left[ c^3(S_{02}) \right] (1 - z\omega) \right]}{E u' \left[ c^3(S_{02}) \right] (1 + r_{12}) (1 + r_{01})} \right] dS_{02}. \quad (A.14)$$
To derive the aggregate gain generated by the change from \( S_{02} \) to \( S_{02} + dS_{02} \), we add (A.12), (A.13), and (A.14). Using the market-clearing equation (24), we find

\[
\sum_{i=1}^{3} dT^i = -\frac{\mathbb{E} \left[ u' \left[ c^3(S_{02}) \right] (1 - z\omega) \right]}{\mathbb{E} u' \left[ c^3(S_{02}) \right] (1 + r_{12}) (1 + r_{01})} dS_{02}.
\]

Therefore, \( S_{02} \) can be optimal only if

\[
\mathbb{E} \left[ u' \left[ c^3(S_{02}) \right] (1 - z\omega) \right] = 0. \tag{A.15}
\]

Eq. (A.15) implies that agent 3 establishes a zero position when allowed to trade in period 0. This means, however, that the aggregate position of agents 1 and 2 is as in the complete-markets equilibrium, i.e., \( S_{02} = S^*_0 \).

\textbf{Proof of Proposition 4:} Agents’ consumption under the optimal maturity structure coincides with that under complete markets. The proposition follows from this observation, (19), (21), and Lemma 2.

\textbf{Proof of Proposition 5:} Eq. (A.1) and CRRA utility imply that under complete markets,

\[
c^2 = \mu^\frac{1}{\gamma} c^3. \tag{A.16}
\]

Substituting (4) and (9) into (A.16), and identifying terms that do not depend on \( r_{12} \), we find

\[
\theta_{02}^2 = \mu^\frac{1}{\gamma} (e_2 + \theta_{02}^3 - S_{02}). \tag{A.17}
\]

Using (8) and \( S_{02}^* = \theta_{02}^1 + \theta_{02}^2 \), we can write (A.17) as

\[
S_{02}^* - \theta_{02}^1 = \mu^\frac{1}{\gamma} (e_2 - S_{02}) \Rightarrow S_{02}^* = \frac{\mu^\frac{1}{\gamma} e_2 + \theta_{02}^1}{\mu^\frac{1}{\gamma} + 1}. \tag{A.18}
\]

If \( \gamma \leq 1 \), then agent 1 is long two-period bonds since \( e^1 < c^1 \). Eq. (A.18) then implies that \( S_{02}^* > 0 \). If \( \gamma \) is large, then agent 1 is short two-period bonds since \( c^1 > \mu^1 \). In the extreme case where \( \alpha = 0 \), \( \theta_{02}^1 \) becomes zero, and (A.18) implies that \( S_{02}^* > 0 \). In the extreme case where \( \alpha = 1 \), \( \mu \) becomes zero, and (A.18) implies that \( S_{02}^* < 0 \).

\textbf{Proof of Proposition 6:} We first show that a decrease in \( \alpha \) raises the aggregate demand \( \theta_{02}^1 + \theta_{02}^2 \) of agents 1 and 2 for two-period bonds if \( \gamma > 1 \), and lowers it if \( \gamma < 1 \). Because of CRRA utility,
the quantities

\[ \phi^1 \equiv \frac{\theta_{02}^1}{\alpha e^0(1 - \tau_0)(1 + r_{01})}, \]
\[ \phi^2 \equiv \frac{\theta_{02}^2}{(1 - \alpha)e^0(1 - \tau_0)(1 + r_{01})}, \]

characterizing the fraction of endowment that agents 1 and 2 invest in two-period bonds, are independent of \( \alpha \). Therefore, a decrease in \( \alpha \) raises \( \theta_{02}^1 + \theta_{02}^2 \) iff \( \phi^2 > \phi^1 \). To compare \( \phi^1 \) and \( \phi^2 \), we substitute \((c^1, c^2)\) from (3) and (4) into (A.1). Using the fact that utility is CRRA, we find

\[ \frac{1 + \phi^1(y - \omega)}{1 + \phi^1(y - \omega)} = \frac{1 + \phi^2(y - \omega)}{1 + \phi^2(y - \omega)} \left( \frac{z}{\bar{z}} \right)^{1 - \gamma}, \]

where \( \bar{z} \) and \( z \) denote respectively the value of \( z \) in state \( r_{12} \) and \( \bar{r}_{12} \). Therefore, the quantity

\[ \frac{[1 + \phi^1(y - \omega)] [1 + \phi^2(y - \omega)] - 1}{[1 + \phi^1(y - \omega)] [1 + \phi^2(y - \omega)]} = \frac{(\phi^2 - \phi^1)(y - \bar{y})}{[1 + \phi^1(y - \omega)] [1 + \phi^2(y - \omega)]} \]

has the same sign as

\[ \left( \frac{\bar{z}}{z} \right)^{1 - \gamma} - 1. \]

Since \( \bar{y} > y \) and \( \bar{z} > z \), the sign of \( \phi^2 - \phi^1 \) is the same as of \( \gamma - 1 \).

The spot rate \( r_{02} \) in the incomplete-markets equilibrium is determined implicitly from the market-clearing equation (24). Since \( \theta_{02}^1 \) and \( \theta_{02}^2 \) are increasing in \( r_{02} \), a decrease in \( \alpha \) lowers \( r_{02} \) iff it raises \( \theta_{02}^1 + \theta_{02}^2 \). Therefore, a decrease in \( \alpha \) lowers \( r_{02} \) if \( \gamma > 1 \), and raises it if \( \gamma < 1 \).

The spot rate \( r_{02} \) in the complete-markets equilibrium is determined implicitly from the market-clearing equation (8). Since \( \alpha \) does not affect the demand \( \theta_{02}^3 \) of agent 3, its impact on \( r_{02} \) has the same sign as under incomplete markets if \( \theta_{02}^3 \) is increasing in \( r_{02} \). The latter follows from a similar argument as in Lemma 3, by writing the first-order condition of agent 3 as

\[ \mathbb{E} [u' [e^2 + A_3 z + \theta_{02}^3(1 - z\omega)] (1 - z\omega)] = 0, \]
where

\[ A_3 \equiv e_1 - (g_0 - c_0 \tau_0)(1 + r_{01}). \]

The effect of \( \alpha \) on \( S_{02}^* \) follows from that on \( r_{02} \) in the complete-markets equilibrium. If \( \gamma > 1 \), then a decrease in \( \alpha \) lowers \( r_{02} \). Therefore, it lowers \( \theta_{02}^3 \) and raises \( S_{02}^* = \theta_{02}^1 + \theta_{02}^2 = S_{02} - \theta_{02}^3 \). The reverse holds if \( \gamma < 1 \).


