OPTIMAL STRATEGY FOR STOCHASTIC PRODUCT ROLLOVER UNDER RISK USING CVAR ANALYSIS

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Abstract

Motivated by many applications such as typical blockbuster product launches, we address in this paper, an inventory/production rollover process between an old and a new product, with a random availability/admissibility date for the new product. The optimization problem consists in finding the phase-in and phase-out dates which minimize a cost minimization objective function. We capture, via a CVaR formulation, the risk phenomenon in the rollover decision making. Then, we provide explicit closed-form expressions for the optimal policies, which can be of three types: Planned Stock-out Rollover, Single-Product Rollover, and Dual-Product Rollover. The analysis led to several managerial insights which are provided in the paper. For instance, we exhibit, first, the impact of risk-aversion on the optimal strategy structure. Then, we show that increasing randomness of the availability date (in the stochastic dominance sense) reinforces the structure of the optimal strategy. We show that the stock-out period is increased in case of optimal Planned Stock-out Rollover and the overlap period is increased for optimal Dual-Product Rollover.
KEYWORDS: Product rollover; Uncertain approval date; Planned stockout rollover (PSR); Single product rollover (SPR); Dual product rollover (DPR); Risk sensitive optimization criterion; Conditional value at risk (CVaR); Stochastic dominance; Stochastic Comparisons

1 INTRODUCTION

Due to rapid technological development and increased variety demanded by consumers, product life cycles have shortened. Therefore, new products have to be introduced and old products phased out more and more frequently. As new product introduction is a source of growth, renewal and competitive advantage, decision makers are facing the issue of how to successfully manage product replacement and optimize the associated supply chain cost trade-offs. In an ideal setting, the optimal rollover strategy is clear: the old product is phased out at the planned introduction date of the new product, and the new product is readily available. Unfortunately, in real-life it is rarely the case. A study of U.S. durable goods companies ([26]) showed that, for various reasons, more than 50 percent of new products failed after being introduced to the market. These poor product launch performances are due to numerous potential random disruption in the process (unexpected logistic or industrial delays, quality problems, inaccurate demand forecasts, unexpected market reactions to the new product announcements, etc...). How to phase in new products while phasing out old ones has become a challenging managerial problem in companies. Obviously, when a company is planning the phase-out of an existing product and the phase-in of a replacement product, classical stochastic production/inventory trade-offs have to be considered. The reason is if production of the existing product is stopped too early, i.e. before the new product is available for the market, the firm will lose sales and customer goodwill. On the other hand, if production of the existing product is stopped too late, the firm will experience an obsolescence cost for the existing product, because demand and/or price would have decreased as this product will be considered ”old generation” by the customers. Furthermore, if the production of the new product is launched too early, the firm will experience an inventory carrying cost until demand picks up. The process of launching a new product in the market
place and the removal of an old one is known as product rollover. In this paper, we focus on three fundamental strategies: planned stockout rollover, single-product rollover and dual-product rollover. An important issue in new product launch management is whether two product generations coexist in the market for a given some time period, i.e. whether the exists an overlapping between successive product generations. In the planned stockout rollover (PSR) strategy, the introduction of the new product is planned in such a way that a stockout phenomenon occurs during the product transition. During this stockout period, no product of any type is available for the market. In the single-product rollover (SPR) strategy, there is a simultaneous introduction of the new product and elimination of the old product, in such a way that at any time there is a unique product generation available in the market. In the dual-product rollover (DPR) strategy, the new product is introduced first and then the old product is phased out. Thus, in this setting, two product generations coexist in the market, for some period of time. The advantage of the DPR strategy, compared to the SPR policy, is to allow for some protection against backorders due to potential random events (delays, quality or market demand level) affecting the planned phasing. The drawback of the DPR strategy is the cost corresponding to the additional inventory in the supply chain. The purpose of this paper is to analyze and characterize the optimality of each type of strategy (PSR, SPR and DPR) for a setting with a stochastic availability date for the new product and taking risk into consideration. Efficient risk measure and optimization is a complex issue, extensively considered in finance research literature. A somewhat recent risk criterion, called conditional value at risk and usually denoted as CVaR, has emerged as exhibiting interesting tractable theoretical properties (see Rockafellar and Uryasev ([30, 31])). Our CVaR model captures the risk issue in the rollover decision making and provides explicit closed-form expressions for the optimal policies. As we consider a quantitative approach, such an analysis requires a performance evaluation model for the supply chain rollover process between two successive products. We provide a newsboy type inventory planning model for the rollover process between two successive products inspired by Hill and Sawaya ([19]) and extended to a risk setting. By solving the associated optimization problem, we obtain the optimality conditions for PSR, SPR or DPR. Furthermore, we show how these different rollover strategies exhibit different
properties w.r.t. the risk. Furthermore, we characterize the influence on the optimal strategy structure of the parameters of the setting, of the size of the randomness and of the manager position with respect to the risk. Our theoretical analysis complements the work of Billington et al. [4] and rigorously show how each strategy (PSO, SPR, DPR) can be optimally associated to the risk aversion and uncertainty level. We formally prove several conjectures concerning optimal structures reported in other papers, that were obtained by empirical research. Also, we investigate the behavior of the optimal rollover policy in response to stochastically larger approval processes.

2 Literature Review

Several papers have addressed the question of efficient management of new product launch, old product termination, or combination of these two processes. A first trend of papers about new product development and launch is mainly of qualitative and descriptive nature (see [23] for a review, encompassing work in marketing, operations management, and engineering design). For instance, Chryssochoidis [10, 11] has studied empirically the whole process in many companies. This research describes the critical causes of delay in international product rollover implementation. Saunders and Jobber [32] identify the different types of strategies when implementing a phase-in and phase-out process. Several papers have addressed the analysis of new product introduction and product rollover processes, under different assumptions. Erhun et al. ([12]) conduct a qualitative study on different drivers affecting product rollovers at Intel Corp., and they develop a framework that guides managers to design and implement appropriate policies taking into consideration rollover risks related to the product, the manufacturing process, the supply chain features, and the managerial policies in a competitive environment. The authors suggest that companies should develop clear strategies for product rollover, in addition to contingency plans in case of failure. They compare and contrast single and dual product rollover strategies. They argue that a single product rollover strategy can be viewed as a high-risk with high return strategy and sensitive to potential random events. On the contrary, the dual product rollover strategy is less risky, but it induces higher inventory costs. Hendricks and Singhal ([18]) have shown by empirical research that delays in new product
introduction decrease the market value of the firm.

The second trend of papers addresses quantitative modeling and optimization of rollover processes. Lim and Tang ([25]) developed a deterministic model that allows the determination of prices of old and new products and the times of phase-in and phase-out of the products. Moreover, they developed marginal cost based conditions to determine when a dual product rollover strategy is more favorable than a single product rollover strategy. Hill and Sawaya ([19]) examine the problem of simultaneously planning the phase-out of the old product and the phase-in of a new product, under an uncertain regulatory approval date for the new product. Under a usual expected profit criterion, they determine the structure of the optimal strategy, which can be linked to the well known newsvendor problem solution.

As our paper deals with a risk-sensitive model, let us refer to the work of Tang ([37]) who provides a concise review of various quantitative models for managing supply chain risks. Most inventory-related papers maximize a predetermined target profit, despite the fact that this may lead to an increased risk. A way to take into account the risk consists of focusing on shortfall, through an absolute bound on the tolerable loss or by setting a bound on the conditional value at risk. Theoretical properties of the CVaR measure of risk has been extensively studied especially in finance (see for example [30, 31]). In inventory theory, some papers have adapted standard results to such risk criterion. For instance, Ozler et al ([29]) utilize Value at Risk (VaR) as a risk measure in a newsboy framework and investigate the multi-product newsboy problem under a VaR constraint. Some papers ([17, 8]) developed closed form solutions due for a CVar newsboy problem.

The structure of this paper is as follows: Section 2 is devoted to the literature while the introduction of the stochastic product rollover problem under consideration is done in section 3. In this same section, we explicit the notations, the optimization criterion, as well as the main assumptions. In section 4, we propose the formulation for the stochastic product rollover problem under risk and we develop the optimality conditions for the three different rollover strategies. Further, we present closed forms solutions and some numerical applications. Finally, in section 5, we propose various managerial insights, conclusions and future research perspectives.
3 The product rollover model

In this section, we will define the product rollover problem and introduce the different notations and assumptions.

3.1 Stochastic rollover process and profit/cost rates

The problem context requires a production plan for the phase-out of an existing product (hereafter called old product, or product 1) and the phase-in of a replacement product (called new product or product 2) under an uncertain admissibility date, denoted $T$, for the new product launch. Typical examples for such admissibility decisions are those of medical devices and pharmaceutical products which cannot be sold until an approval body grants permission. Two decision variables have to be fixed in such a rollover process: $t_1$, the date the firm plans to phase-out product 1 and $t_2$, the date product 2 is planned to be ready and available for the market. Product 1 continues to be sold until inventory is depleted or until it is replaced by the new approved product. The manufacturing and procurement lead times are assumed to be large, thus making it necessary to commit to the planning dates before the random approval date is revealed. Therefore, the decision process relies exclusively on the probability distribution of this date $T$. Such large procurement/manufacturing/distribution lead-times are frequent in practice. During its regular commercial life span, each product is assumed to have a specific deterministic constant demand rate, namely $d_1$ for product 1 and $d_2$ for product 2. A channel inventory is needed to support each product in the market, which induces carrying inventory cost rates $c_{h,1}$ and $c_{h,2}$. During the commercial life span, the contribution-to-profit rate for product $i$, is defined as

$$m_i = d_i(p_i - c_i) - c_{h,i}, \quad (i = 1, 2),$$

with $p_i$ the selling price and $c_i$ the production cost.

3.1.1 Cost rates model for the PSO strategy

Furthermore, in our considered random setting, the profit and cost structure depends on the relative values of $t_1$, $t_2$ and $T$. Indeed, if the strategy $t_1 \leq t_2$ is chosen, the
structure of the profit and cost rates, defined over an infinite time horizon, is given in Figure 1,

![Figure 1: the profit rates when $t_1 \leq t_2$](image)

As shown in Figure 1, there are three main cases to be considered. First if $T \leq t_1$, the profit rate is $m_1$ over the time interval $[0,T]$. Then, over $[T,t_1[$, product 2 is admissible, but not yet physically available in the supply chain. As the market is assumed to be informed that product 2 will soon substitute product 1, profit rate of product 1 changes from $m_1$ to $m'_1$ as long as product 1 is available, i.e. over $[T,t_1[$. This contribution rate $m'_1$ is formally given by

$$m'_1 = d'_1(p'_1 - c_1) - c_{h,1}. \quad (2)$$

Then, over the interval $[t_1,t_2]$, when product 1 is sold out, shortages occur until product 2 delivery date $t_2$, at a corresponding shortage cost rate $g$. Once product 2 is available, at $t_2$, the profit rate becomes $m_2$ over the remaining time horizon $[t_2,\infty[.$ Then, when $t_1 \leq T \leq t_2$, the rates are similar to the first case, except over $[0,t_1]$, where the profit rate is $m_1$. Finally, if $t_2 \leq T$, the profit/cost rates are similar to the previous situation, except over the interval $[t_2,T]$, where product 2 is physically available in the supply chain, but still not admissible. A shortage cost rate $g$ occurs until product 2 is admissible. In addition, an inventory cost rate $c_{h,2}$ associated with the physical inventory of product 2 is incurred.

### 3.1.2 Cost rates model for the DPR strategy

If the strategy $t_2 \leq t_1$ is chosen, the structure of the costs and profit rates is given in Figure 2.
First, let us consider the instance where $T < t_2$. The profit rate is $m_1$ over the time interval $[0, T]$ and $m'_1$ over $[T, t_2]$. Then, over the time interval $[t_2, t_1]$, as product 2 is admissible and physically available, it is sold with a profit rate $m_2$. However, in the current setting, it is assumed that the firm immediately scraps, at a cost rate $s_1$, all the remaining inventory of product 1 when an approved product 2 is available for sale, i.e. over the time interval $[T, t_1]$. This is justified by the higher margins for product 2 and by the need to maintain brand equity as a leading-edge provider. Finally, over the remaining time horizon $[t_1, \infty[$, the profit rate resumes to $m_2$.

When $t_2 \leq T \leq t_1$, the profit rate is $m_1$ over $[0, t_2]$. Then over the interval $[t_2, T]$, the profit rate is still $m_1$, but as product 2 is physically available in the supply chain, but not admissible for sale, an inventory cost rate $c_{h,2}$ is incurred. Over the remaining horizon starting at $T$, product 2 is sold with a profit rate $m_2$. In the time interval $[T, t_1]$, product 1 is scrapped at a cost rate $s_1$.

Finally, when $t_1 \leq T$, the profit rate is $m_1$ over $[0, t_2]$. Then, over the interval $[t_2, t_1]$ the profit rate is still $m_1$, but an inventory cost rate $c_{h,2}$ has to be incurred. Over $[t_1, T]$, product 1 is phased-out and product 2 is not yet admissible, thus creating a shortage cost rate $g$. Over the remaining time horizon $[T, \infty[$, the profit rate reverts to $m_2$.

### 3.2 Model Notations

The main notations used in the product rollover model are summarized here. For each product type $i \in \{1, 2\}$, we define

$c_i$ : the unit cost of product $i$,
$p_i$ : the unit price of product $i$,
$d_i$ : the demand rate of product $i$,
$c_{h,i}$ : the carrying cost rate of product $i$,
$m_i$ : the contribution-to-profit rate of product $i$, defined as $m_i = d_i(p_i - c_i) - c_{h,i}$.

Furthermore, we define

$g$ : the lost of goodwill rate when the firm has none of the products for sale,
$m'_1$ : the new contribution-to-profit rate of product 1 after the admissibility of product 2 is granted,
$s_1$ : the scrap cost rate for product 1.

Furthermore, we denote

$T$ : the random approval date for product 2. This random variable has a density probability function $f(\cdot)$ and an associated probability distribution function $F(\cdot)$, both defined over $[0, \infty[$.

The decision variables are

$t_1$ : the planned run-out date for product 1,
$t_2$ : the planned availability date for product 2.

Further, we note

$[Y]^+ := \max(Y, 0)$.

### 3.3 The Performance Criterion for the Rollover Problem

Here, we consider a performance criterion defined as the difference between the profit under complete information about admissibility date, and the profit when the admissibility date is random and known exclusively through its probability distribution.

This performance criterion is defined as follows: when the admissibility date is known before the decisions $t_1$ and $t_2$ are made, as depicted in Figure 3,

![Figure 3: Full information case](image-url)

Figure 3: Full information case
the optimal strategy is clearly given by: \( t_1 = t_2 = T \), i.e., product 1 is phased out at the planned introduction date of product 2, corresponding to the admissibility date. We can see that over the time interval \([0, T]\), the profit rate is \( m_1 \), while on the remaining horizon \([T, \infty]\), the profit rate is \( m_2 \).

In order to characterize the impact of randomness on the rollover process, we consider an infinite horizon objective function defined as the difference between the perfect information cost rate function (Figure 3) and the cost rates functions with imperfect information (Figures 1 and 2). This difference can be interpreted as the loss caused by the randomness plaguing the availability date \( T \). Formally, according to the description given above, these loss functions are piecewise linear and exhibit different structures depending on the relative values of the decision variables \( t_1 \) and \( t_2 \). If a strategy with \( t_1 \leq t_2 \) is chosen, the loss rate function is denoted as \( L_1(t_1, t_2; T) \) and can be expressed as:

\[
L_1(t_1, t_2; T) = \begin{cases} 
(m'_1 - m_2)(t_1 - T) - (m_2 + g)(t_2 - t_1) & \text{if } 0 \leq T \leq t_1, \\
-(g + m_1)(T - t_1) - (g + m_2)(t_2 - T) & \text{if } t_1 \leq T \leq t_2, \\
-(g + m_1)(t_2 - t_1) - (g + m_1 + c_{h,2})(T - t_2) & \text{if } t_2 \leq T.
\end{cases}
\]

It is rewritten as

\[
L_1(t_1, t_2, T) = (m_1 + g)[T - t_1]^+ - (m + m'_1)[t_1 - T]^+ \\
+ c_{h,2}[T - t_2]^+ + (m_2 + g)[t_2 - T]^+.
\] (3)

If a strategy \( t_2 \leq t_1 \) is chosen, the loss function is denoted as \( L_2(t_1, t_2; T) \) and is given by

\[
L_2(t_1, t_2; T) = \begin{cases} 
(m'_1 - m_2)(t_2 - T) - s_1(t_2 - t_1) & \text{if } 0 \leq T \leq t_1, \\
-c_{h,2}(T - t_2) - s_1(t_1 - T) & \text{if } t_1 \leq T \leq t_2, \\
-c_{h,2}(t_2 - t_1) - (g + m_1)(T - t_1) & \text{if } t_2 \leq T.
\end{cases}
\]

It is rewritten as

\[
L_2(t_1, t_2, T) = (m_2 - m'_1 - s_1)[t_2 - T]^+ + c_{h,2}[T - t_2]^+ \\
+ (m_1 + g)[T - t_1]^+ + s_1[t_1 - T]^+.
\] (4)
If we formally introduce the two regions, \( R_1 = \{(t_1, t_2) \in \mathbb{R}_+^2 : t_1 \leq t_2 \} \) and \( R_2 = \{(t_1, t_2) \in \mathbb{R}_+^2 : t_1 \geq t_2 \} \), the piecewise loss rate function can be rewritten as

\[
L(t_1, t_2, T) = L_i(t_1, t_2, T) \quad \text{if} \quad (t_1, t_2) \in R_i \quad (i = 1, 2).
\]  

(5)

On \( R_b \), defined as the boundary between regions \( R_1 \) and \( R_2 \), i.e. for \( R_b = \{(t_1, t_2) \in \mathbb{R}_+^2 : t_1 = t_2 \} \), the expression of the objective function is obtained from (3) and (4) as

\[
L_b(t, T) = (m_2 - m'_1)[t - T]^+ + (m_1 + g + c h_2)[T - t]^+.
\]  

(6)

3.4 Assumptions for Problem Parameters

In order to guarantee the significance of the model, it is necessary to introduce some assumptions for the different problem parameters. First the contribution-to-profit rate for the products during regular sale period is positive, i.e.

\[ m_1, m_2 > 0. \]  

(7)

Furthermore, for product 1, the contribution-to-profit rate during regular sale period is strictly greater than the (possibly negative) contribution to profit rate after product 2 is available, i.e.

\[ m_1 > m'_1. \]  

(8)

In order to avoid cases for which it would be optimal to delay infinitely the new product launch, we assume

\[ m_2 > m'_1. \]  

(9)

Finally, as for any classical inventory problem, we assume,

\[ g, c h_2, s_1 > 0. \]  

(10)

4 CVar Formulation of the Rollover Problem

In order to introduce the impact of risk aversion in the decision process, we consider our problem in a CVaR-minimization context (see [30, 31]). First, we introduce CVaR formulation of the rollover problem. Then, we establish the corresponding expression of the analytical optimal solutions. Finally, we analyze the impact of the risk-aversion on the selected rollover policies.
4.1 Conditional Value at Risk formulation

For a given probability distribution $F(\cdot)$ associated with the random date $T$, let us denote the probability distribution function of the loss function $L(t_1, t_2; T)$ by

$$L_F(\eta|t_1, t_2) = Pr\{ L(t_1, t_2, T) \leq \eta \}. \quad (11)$$

For any $\beta \in [0, 1)$, we define the $\beta$-VaR of this distribution by

$$\alpha_\beta(t_1, t_2) = \min \{ \alpha | L_F(\alpha|t_1, t_2) \geq \beta \}. \quad (12)$$

It is now possible to introduce the $\beta$-tail distribution function to focus on the upper tail part of the loss distribution as

$$L_{F,\beta}(\eta|t_1, t_2) = \begin{cases} 0 & \text{for } \eta < \alpha_\beta(t_1, t_2), \\ \frac{\beta(\eta|t_1, t_2) - \alpha_\beta(t_1, t_2)}{1 - \beta} & \text{for } \eta \geq \alpha_\beta(t_1, t_2). \end{cases} \quad (13)$$

Using the expectation operator $E_\beta[\cdot]$ under the $\beta$-tail distribution $L_{F,\beta}(\cdot|\cdot)$, we define the $\beta$-conditional value-at-risk of the loss $L(t_1, t_2; T)$ by

$$E_\beta[L(t_1, t_2; T)]. \quad (14)$$

Finding the optimal rollover structure and the optimal values of the phase-in and phase-out dates which minimize the a CVaR cost criterion requires solving the following optimization problem

$$\min_{(t_1, t_2) \in \mathbb{R}_+^2} E_\beta[L(t_1, t_2; T)]. \quad (15)$$

According to [30, 31], the minimization of $E_\beta[L(t_1, t_2; T)]$ with respect to the decision variables $t_1$ and $t_2$ amounts to the minimization of the following auxiliary function

$$l_\beta(t_1, t_2, \alpha) := \alpha + \frac{1}{1 - \beta} E_F[L(t_1, t_2; T) - \alpha^+]. \quad (16)$$

$l_\beta(t_1, t_2, \alpha)$ is known to be convex with respect to $\alpha$ (see [30, 31]). According to the specific structure of the loss function (3)-(4), it is natural to associate to (16) a pair of auxiliary functions for $i = 1, 2$, defined as follows

$$l_{\beta,i}(t_1, t_2, \alpha) = \left\{ \alpha + \frac{1}{1 - \beta} E_F[L_i(t_1, t_2, T) - \alpha^+] \right\}, \quad (17)$$

and an auxiliary function on the boundary,

$$l_{\beta,b}(t, \alpha) = \left\{ \alpha + \frac{1}{1 - \beta} E_F[L_b(t, T) - \alpha^+] \right\}. \quad (18)$$
4.2 Convexity and structure of the optimal solutions

The optimal solution structure is essentially determined by concavity and convexity characteristics of the functions (17)-(18) over the regions $R_1$ and $R_2$.

**Property 1:** Under assumption (8), the CVaR loss function $l_{β,1}(\cdot,\cdot,\cdot)$ is strictly jointly convex on $R_3^\beta$.

*Proof.* This property stems from convexity of the loss function $L_1(\cdot,\cdot)$, combined with a specific property of the CVar formulation (see Appendix A).

**Corollary**: The boundary loss function $l_{β,b}(\cdot,\cdot)$ is strictly jointly convex on $R_2^\beta$.

*Proof.* The proof is direct as $l_{β,b}(\cdot)$ can be viewed as the intersection of $l_{β,1}(\alpha,t_1,t_2)$ by an hyperplane defined by $t_1 = t_2$ (with $t_1,t_2 \geq 0$).

**Property 2:** Under the assumption $(m_2 - m_1' - s_1 + c_{h,2}) > 0$, the CVaR-loss function $l_{β,2}(\cdot,\cdot,\cdot)$ is strictly jointly convex over $R_3^\beta$. Otherwise, $l_{β,2}(t_1,\cdot,T)$ is a strictly decreasing function.

*Proof.* This property stems from convexity of the loss function $L_2(\cdot,\cdot)$, combined with a specific property of the CVar formulation (see Appendix A).

The structure of the optimal strategy depends on the combination of the convexity properties of the functions $l_{β,1}(\cdot,\cdot,\cdot)$ and $l_{β,2}(\cdot,\cdot,\cdot)$ and on the location of their minimum. As defined in the introduction, we observe three types of strategies: planned stock-out rollover, single product rollover, and dual product rollover. If one denotes the respective minimum of the $l_{β,i}(\cdot,\cdot,\cdot)$ functions over $R_3^\beta$ as $(α_{β,i}^*,t_{β,1,i}^*,t_{β,2,i}^*)$, the optimal strategy structure is displayed in Table 1,
Table 1: Convexity properties and structure of the optimal rollover strategy

In the next sections, we show how the cost/profit parameters fix the optimal strategy structure.

4.3 First-order Optimality Conditions

The minimum of the auxiliary loss functions \( l_{\beta,1}(\cdot, \cdot, \cdot) \), \( l_{\beta,2}(\cdot, \cdot, \cdot) \) and \( l_{\beta,2}(\cdot, \cdot) \) are characterized in the following properties by classical first-order conditions.

**Property 3.a**: Consider the setting \( m_2 \geq m_1 > m_1' \). Under the assumption

\[
\frac{m_1 + g}{m_1 - m_1'} < \frac{c_{h,2}}{m_2 + c_{h,2} + g},
\]

the first-order conditions solutions

\[
l_{\beta,1}(\cdot, \cdot, \cdot) \quad \text{properties:} \quad (t_{\beta,1,1}', t_{\beta,1,2}') \notin R_2
\]

\[
l_{\beta,2}(\cdot, \cdot, \cdot) \quad \text{properties:} \quad (t_{\beta,2,1}', t_{\beta,2,2}') \in R_2
\]

\[
\text{Global Optimal Solution} \quad \text{Convex} \quad (t_{\beta,1,1}', t_{\beta,1,2}') \in R_2
\]

\[
\text{Optimal Strategy Structure} \quad \text{Strictly decreasing w.r.t.} \ t_2 \quad \text{or convex Convex} \quad (t_{\beta,2,1}', t_{\beta,2,2}') \in R_2
\]

\[
\text{Planned Stockout} \quad \downarrow \quad \text{Optimal Strategy Structure}
\]

\[
\text{Single Product Rollover} \quad \text{Dual Product Rollover}
\]

If condition (19) is not satisfied, there is no finite minimum in \( R_1 \) for \( l_{\beta,1}(\cdot, \cdot, \cdot) \).

**Proof.** See Appendix B-1.

**Property 3.b**: Consider the setting \( m_1 > m_2 \geq m_1' \). Under the assumptions

\[
m_1' < -g,
\]

\[
m_2 > m_1 - c_{h,2},
\]

\[
(19)
\]

\[
(20)
\]

\[
(21)
\]

\[
(22)
\]

\[
(23)
\]
and for
\[ \beta \in [\beta_{1,mf}, 1 + \frac{m_2 - m_1}{c_{h,2}}], \]  
with
\[ \beta_{1,mf} = \begin{cases} 0 & \text{if } m_1 - m'_1 > m_2 + g + c_{h,2}, \\
\frac{(m_1 + g)(m_2 + g + c_{h,2} - m_1 + m'_1)}{(m_1 + g)(m_2 + g) + c_{h,2} - m_1 + \beta} & \text{if } m_1 - m'_1 \leq m_2 + g + c_{h,2}. \end{cases} \]  

the first-order conditions solutions
\[ r_{\beta,1,1}^* = \left( \frac{m_2 - m'_1}{m_1 - m'_1} \right)^{F^{-1}} \left( \frac{(m_1 + g)(1 - \beta)}{m_1 - m'_1} \right) + \left( \frac{m_3 - m_2}{m_3 - m'_1} \right)^{F^{-1}} \left( \frac{m_3 - m'_1 + g(1 - \beta)}{m_3 - m'_1} \right), \]  
\[ r_{\beta,1,2}^* = F^{-1} \left( \frac{m_1 + g + c_{h,2} \beta}{m_2 + g + c_{h,2}} \right). \]  

Correspond to the unique finite minimum of the CVaR loss function \( l_{\beta,1}(\cdot, \cdot, \cdot) \) in \( \mathbb{R}_1 \).

If condition (22), (23), or (24) is not satisfied, there is no finite minimum in \( \mathbb{R}_1 \).

**Proof.** See Appendix B-2.

**Property 4.a:** Consider the setting \( c_{h,2} > s_1 \). Under the assumptions
\[ m_2 - m'_1 - s_1 \geq 0, \]  
\[ \frac{m_1 + g}{m_1 + g + s_1} > \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}, \]  
the first-order conditions solutions
\[ r_{\beta,2,1}^* = F^{-1} \left( \frac{m_1 + g + s_1 \beta}{m_1 + g + s_1} \right), \]  
\[ r_{\beta,2,2}^* = \left( \frac{m_2 - m'_1}{m_2 - m'_1 - s_1 + c_{h,2}} \right)^{F^{-1}} \left( \frac{c_{h,2} (1 - \beta)}{m_2 - m'_1 - s_1 + c_{h,2}} \right) + \left( \frac{c_{h,2} - s_1}{m_2 - m'_1 - s_1 + c_{h,2}} \right)^{F^{-1}} \left( \frac{c_{h,2} + \beta (m_2 - m'_1 - s_1)}{m_2 - m'_1 - s_1 + c_{h,2}} \right) \]  

Correspond to the unique finite minimum of the CVaR loss function \( l_{\beta,2}(\cdot, \cdot, \cdot) \) in \( \mathbb{R}_2 \).

If condition (28) or (29) is not satisfied, there is no finite minimum in \( \mathbb{R}_2 \).

**Proof.** See Appendix B-3.

**Property 4.b:** Consider the setting \( c_{h,2} \leq s_1 \). Under the assumptions
\[ m_2 - m'_1 - s_1 + c_{h,2} > 0, \]  
\[ m_2 - m'_1 - s_1 < c_{h,2}, \]  
\[ \frac{m_1 + g}{m_1 + g + s_1} > \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}, \]  

and for \( \beta \) values satisfying
\[ \beta > \frac{m_2 - m'_1 - s_1}{c_{h,2}}, \]  

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correspond to the unique finite minimum of the CVaR loss function $l_{\beta,2}(\cdot,\cdot,\cdot)$ in $\mathbb{R}_2$. If condition (32), (33), (34) or (35) is not satisfied, there is no finite minimum in $\mathbb{R}_2$ for $l_{\beta,2}(\cdot,\cdot,\cdot)$.

**Proof.** See Appendix B-4.

**Property 5:** Under assumption (9), the boundary loss function $l_{\beta,b}(\cdot,\cdot)$ has a unique finite minimum over $\mathbb{R}^+$ corresponding to

$$t^*_b,FOC = \left( \frac{m_2 - m_1'}{m_2 - m_1' + m_1 + c_{h,2} + g} \right) F^{-1} \left( \frac{m_1 + c_{h,2} + g (1 - \beta)}{m_2 - m_1' + m_1 + c_{h,2} + g} \right)$$

$$+ \left( \frac{m_1 + c_{h,2} + g}{m_2 - m_1' + m_1 + c_{h,2} + g} \right) F^{-1} \left( \frac{m_1 + c_{h,2} + g + \beta (m_2 - m_1')}{m_2 - m_1' + m_1 + c_{h,2} + g} \right). \quad (38)$$

**Proof.** See Appendix B-5.

### 4.4 Optimal Product Rollover Strategies

The above properties are used in Table 2, which exhibits the optimal product rollover strategy structure depending on the critical constraints on the parameters.

#### 4.4.1 Parameters and associated optimal strategy structure.

First, it can be observed in Table 2 that the different strategy structures (PSO, DPR and SPR) can apply to the four main cases, namely $(m_2 \geq m_1 > m_1'; c_{h,2} \geq s_1)$, $(m_2 \geq m_1 > m_1'; c_{h,2} < s_1)$, $(m_1 > m_2 \geq m_1'; c_{h,2} \geq s_1)$ and finally $(m_1 > m_2 \geq m_1'; c_{h,2} < s_1)$. Indeed, this shows that the structure of the optimal rollover strategy depends simultaneously, and in a complex manner, on all the parameters. Furthermore, for most cases (highlighted in grey) the optimal strategy structure is independent from the probability distribution $F(\cdot)$.

Second, the PSO strategy can be optimal exclusively for products with significantly negative margins $m_1'$ (i.e. when the market or the price of product 1 collapses once
\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Condition} & \text{Action} & \text{Comment} & \text{Comment} & \text{Comment} \\
\hline
m_2 \geq m_1 & m_3 > m_1 & \text{m_2 > m_1 - c_{h,2}} & m_1 < m_1' & \text{m_2 > m_1 - c_{h,2}} \\
\hline
m_2 \geq m_3 & m_1 \geq m_1' & \text{m_1 > m_1'} & m_1 \geq m_1' & \text{m_1 > m_1'} \\
\hline
m_2 \geq m_3 & m_1 \geq m_1' & \text{m_1 > m_1'} & m_1 \geq m_1' & \text{m_1 > m_1'} \\
\hline
\end{array}
\]

Table 2: Optimal Rollover Strategy Structure
product 2 is approved). If the margin $m_1'$ remains positive, then the optimal strategy should be SPR or DPR.

Third, high scrap cost rates $s_1$ exclude DPR, while for high holding cost rates $c_{h,2}$, PSO appears to be attractive.

Fourth, if the margins of products 1 and 2 are similar, i.e. $m_1 \approx m_1' \approx m_2$, then the optimal product rollover strategy is SPR.

4.4.2 Risk Aversion and Optimal Strategy Structure

As shown in above subsection, the optimal strategy structure depends simultaneously on the different parameters of the problem, on the probability distribution $F(\cdot)$, as well as on the risk aversion defined through $\beta$. While, it is tedious to find explicit necessary and sufficient optimality conditions for each type of rollover strategy based on these different factors, the specific impact of risk aversion over the optimal strategy structure can be analyzed. As mentioned previously, $\beta$ reflects the degree of risk aversion for the planner: the larger $\beta$ is, the more risk averse the planner is. By using the above properties of the CVar-loss functions, Tables 3 and 4 are developed based on two significantly different situations: low risk aversion ($\beta$ value close to zero) and high risk aversion ($\beta$ value close to 1).

4.4.3 The Low Risk Aversion Case

For this case (depicted in Table 3), the analysis is done with $\beta$ close to zero.
\[
\begin{align*}
\text{Table 3: Optimal Rollover Policy Structure under low risk aversion}
\end{align*}
\]
It is worth noting that the optimal strategy structure is dependent on the decision maker risk aversion. As a matter of fact, by changing \( \beta \) value, it can be seen in Table 3 and Table 4 that for some parameters combinations the structure of the optimal strategy change for low or high \( \beta \) values. Even if the exact analysis of the impact of the risk aversion factor \( \beta \) requires several cases (as depicted in Table 3 and Table 4), it can be noted that lowering the \( \beta \) value induces optimal strategy structure change toward PSO (two cases), SPR (four cases) and DPR (two cases). Increasing the \( \beta \) value induces optimal strategy structure change toward SPR (four cases) and DPR (four cases). All these changes are independent from the probability distribution \( F(\cdot) \).

This short analysis shows that a conjecture, empirically given in ([4]), arguing single rollover to be a high-risk, high-return strategy while dual rollover to be less risky, has to be taken with care in practice. Clearly, the structure of the optimal strategy simultaneously depends on the costs structure and on the risk aversion.
Table 4: Optimal Rollover Policy Structure under high risk aversion
5 Impact of Uncertainty

In this section, we study the variation of the optimal product rollover policy structure and of the associated optimal cost, when increasing stochasticity of the random admissibility date $T$. We have shown in the preceding section that risk aversion level can change the structure of the optimal product rollover strategy.

Billington et al., in their paper about efficient rollover strategies [4], present SPR as a high risk strategy, suited to situations with low uncertainty and DPR as a low risk strategy, suited to situations with a higher uncertainty. Increasing uncertainty level reinforces the rollover policy type (i.e. increase the overlap between $t_1^*$ and $t_2^*$ (positive or negative)), while the decision maker risk aversion can change the optimal strategy structure (i.e. change the $t_1^*$, $t_2^*$ ordering). The main motivation of this section consists in theoretically analyzing this conjecture by Billington et al. (see [4]) claiming that when the variability of the new product admissibility date increases, then basically the overlap, i.e. the positive or negative gap between $t_1$ and $t_2$, in the optimal solution has to increase too. This property is theoretically known as a dispersive ordering property. Here, we formally give the conditions guaranteeing this conjecture. It can be seen that the impact of the variability of the admissibility date is threefold: impact on the optimal global cost, impact on the optimal value of each of the two decision variables and impact on the structure of the optimal strategy. In such an analysis, the key element is the formal definition of variability or stochasticity increase between a pair of probability distribution functions.

In order to assess the variability effects on the considered model, we conduct a stochastic comparison between two rollover processes. We consider two rollover processes $i = 1, 2$, with approval dates $T_i$, known through their cumulative probability distribution functions $F_i$. Here, we focus on the variability effects of $T_i$ and thus we assume that the admissibility dates have equal means, $E[T_1] = E[T_2]$. In order to compare the variabilities of the pair of random variables $T_1$ and $T_2$, we will have to define as stochastic ordering criterion.

First, we focus on the changes of the optimal solution values and on the change of the optimal cost when the problem variability increases. These changes can be theoretically characterized along the lines of ([36, 39]). In order to define the concept
of variability increase, we consider a stochastic ordering based on a comparison of the spread of the probability density functions.

Second, we focus on the change of the optimal product rollover strategy structure, namely the change of the overlap size associated with the optimal policies. We recall that in case of a positive overlap (corresponding to DPR), the pair of products are simultaneously available on the market during some time period, while in case of a negative overlap (corresponding to PSO), no product is available on the market. To do so, we need to use a more restrictive stochastic ordering assumption, known as dispersive ordering condition [20, 21, 35, 39].

5.1 Impact of Uncertainty on the average loss and on the optimal decisions

5.1.1 The Considered Stochastic Ordering

We consider here a usual stochastic ordering, based on the shapes of the distribution functions and defined as follows. Let \( u(t) \) be a real function defined on an ordered set \( U \) of the real line and let \( S(u) \) be the number of sign changes of \( u(t) \) when \( t \) ranges over the entire set \( U \).

**Definition.** Consider two random variables \( T_1 \) and \( T_2 \) with the same mean, i.e. \( E[T_1] = E[T_2] \), having probability distributions \( F_1(\cdot) \) and \( F_2(\cdot) \) with densities \( f_1(\cdot) \) and \( f_2(\cdot) \). We say that \( T_1 \) is more variable than \( T_2 \), denoted \( T_1 \geq_{\text{var}} T_2 \), if

\[
S(f_1 - f_2) = 2 \text{ with sign sequence } +, -, +. \tag{39}
\]

That is, \( f_1(\cdot) \) crosses \( f_2(\cdot) \) exactly twice, first from above and then from below. It is known (see [39]), that when \( E[T_1] = E[T_2] \), condition (39) implies that

\[
F_1(x) \leq F_2(x) \text{ for all } x \quad \text{and} \quad E[h(T_1)] \geq E[h(T_2)] \tag{40}
\]

for all nondecreasing functions \( h(\cdot) \). Observe that condition (39) also implies that

\[
S(F_1 - F_2) = 1 \tag{41}
\]

with sign sequence \(+, -, +\), in other words, \( F_1(\cdot) \) crosses \( F_2(\cdot) \) exactly once, and the cross is from above. Furthermore, it is also known (see [39]) that equation (41) implies

\[
\int_{-\infty}^{t} (F_1(x) - F_2(x)) \, dx \leq 0. \tag{42}
\]
Examples of pairs of distributions satisfying condition (41) are given in reference [36] and include a large number of important standard unimodal densities arising in statistical applications, as seen from the following pairs ($i = 1, 2$):
- $f_i(\cdot)$ are Gamma (Weibull) with shape parameter $\eta_1, \eta_2$, with $\eta_2 < \eta_1$;
- $f_i(\cdot)$ are Uniform $(a_i, b_i)$, with $a_1 < a_2$, $b_1 > b_2$, but $a_1 + b_1 = a_2 + b_2$;
- $F_i(\cdot)$ are Gaussian with parameters $\mu_i$ and $\sigma_i$, with $\mu_1 = \mu_2$ and $\sigma_2 < \sigma_1$;
- $f_i(\cdot)$ are truncated Gaussian with parameters $\mu_i$ and $\sigma_i$, with $\mu_1 = \mu_2 \gg 0$ and $\sigma_2 < \sigma_1$;
- $f_1(\cdot)$ is decreasing (e.g., exponential) and $f_2(\cdot)$ is Uniform.

5.1.2 Impact of variability on the decision variables

We now present our results regarding the effect of approval date variability on the optimal times.

Property 6. If $T_1 \geq \text{var } T_2$, then there exists a critical number $\theta_{F_1,F_2}$ such that

\[
\begin{align*}
F_1^{-1}(r) & \leq F_2^{-1}(r) \quad \text{if } 0 \leq r \leq \theta_{F_1,F_2}, \\
F_1^{-1}(r) & \geq F_2^{-1}(r) \quad \text{if } \theta_{F_1,F_2} < r \leq 1.
\end{align*}
\]

Proof : the proof follows reference [36]. Condition $T_1 \geq \text{var } T_2$ implies that $F_1(\cdot)$ crosses $F_2(\cdot)$ exactly once for $x = x^*$ (i.e. one has $F_1(x^*) = F_2(x^*)$), and the cross is from above. That means that there exists $x^*$ such that, for $0 < x < x^*$, $F_1(x)$ is at least as large as $F_2(x)$ and for $x < x^*$, $F_1(x)$ is at most as large as $F_2(x)$. Setting $\theta_{F_1,F_2} = F_1(x^*) = F_2(x^*)$, the results regarding the order of $F_i^{-1}(r)$ are immediate.

A direct application of the above proposition is the following corollary.

Corollary 6.1 If $T_1 \geq \text{var } T_2$, the value of the different optimal solutions corresponding to the first order conditions (20)-(21), (26)-(27), (30)-(31), (36)-(37) and (38) can increase or decrease, depending on the parameter values and on the probability distributions $F_1(\cdot)$ and $F_2(\cdot)$.

Proof : Let us denote the distribution dependence of the different first order solutions respectively corresponding to equations (20)-(21), (26)-(27), (30)-(31), (36)-(37) and (38), as $t^{*,FOC}(F)$, which can be formally expressed as a linear combination of
quantiles as follows,
\[ t^{*, FOC}(F) = \gamma F^{-1}(r_1) + (1 - \gamma) F^{-1}(r_2), \quad \text{with} \quad 0 \leq \gamma \leq 1, \quad (43) \]
with the parameters \( \gamma, r_1 \) and \( r_2 \) extensively defined in equations (20)-(21), (26)-(27), (30)-(31), (36)-(37) and (38). Now, as the optimal solutions \( t^{*, FOC}(F) \) are convex combinations of quantiles, one finds the following orderings
\[
\begin{cases}
    t^{*, FOC}(F_1) \leq t^{*, FOC}(F_2) & \text{if } 0 \leq r_1, r_2 \leq \theta_{F_1,F_2}, \\
    t^{*, FOC}(F_1) \geq t^{*, FOC}(F_2) & \text{if } \theta_{F_1,F_2} \leq r_1, r_2 \leq 1,
\end{cases}
\]
while the ordering is undefined when \( 0 \leq r_1 \leq \theta_{F_1,F_2} \leq r_2 \) or \( 0 \leq r_2 \leq \theta_{F_1,F_2} \leq r_1 \).

This shows that, for increasingly variable distributions, the sign of the change of the optimal solutions is not straightforward and depends on the order relationship between the threshold \( \theta_{F_1,F_2} \) and the different ratios defining the optimal solution values.

5.1.3 **Impact of variability on the average loss**

The following proposition establishes the intuitive result that increasing variability increases the expected loss.

**Property 7.** If \( T_1 \geq_{var} T_2 \), then
\[
\min_{(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+} E_{F_1}[L(t_1, t_2, T)] \leq \min_{(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+} E_{F_2}[L(t_1, t_2, T)]. \quad (44)
\]

**Proof.** This result is obtain by applying the stochastic ordering assumption to the expected loss expression (See Appendix E).

5.2 **Impact of Uncertainty on structure of the optimal product rollover strategy**

5.2.1 **Stochastic Ordering Definitions**

This subsection analyzes the impact of uncertainty on the structure of the product rollover optimal strategy, e.g. on the size of the overlap between the planning. The analysis, focused on the difference between the optimal decisions for \( t_1 \) and \( t_2 \), and not their individual values, relies on another class of stochastic ordering, called dispersive ordering, as defined below.
**Definition.** Consider two random variables $T_1$ and $T_2$ with same mean $E[T_1] = E[T_2]$, having distributions $F_1(\cdot)$ and $F_2(\cdot)$ with densities $f_1(\cdot)$ and $f_2(\cdot)$. $T_1$ is said to be less dispersed than $T_2$, denoted by $T_1 <_{\text{disp}} T_2$, if

$$F_2^{-1}(a) - F_2^{-1}(b) < F_1^{-1}(a) - F_1^{-1}(b), \quad \text{whenever } 0 < a < b < 1. \quad (45)$$

This means that the difference between any pair of quantiles of $F_2(\cdot)$ is smaller than the difference between the corresponding quantiles of $F_1(\cdot)$. It is well known that this condition is more restrictive than (39) or (41). Examples include a large number of important standard unimodal densities (see [20]) as Gamma densities, Uniforms, Gaussians, truncated Gaussians, and others.

### 5.2.2 Impact of variability on the overlap of the optimal rollover structure

**Property 8.** We denote the optimal decisions associated with a probability distribution $F(\cdot)$ by $t_1^*(F)$ and $t_2^*(F)$. If $T_1 \geq_{\text{disp}} T_2$, then if the optimal strategy is DPR, the overlap period increases, one has

$$t_1^*(F_1) - t_2^*(F_1) \geq t_1^*(F_2) - t_2^*(F_2), \quad (46)$$

if the strategy is PSO, the stockout period increases, one has

$$t_2^*(F_1) - t_1^*(F_1) \geq t_2^*(F_2) - t_1^*(F_2). \quad (47)$$

**Proof:** This is a direct application of the stochastic ordering to the optimality conditions. First, we note that for a given probability distribution $F(\cdot)$, the optimal decisions $t_1^*(F)$ and $t_2^*(F)$ are convex combination of quantiles, which can be denoted as follows,

$$t_1^*(F) = \gamma F^{-1}(r_{1,1}) + (1 - \gamma) F^{-1}(r_{2,1}), \quad \text{with} \quad 0 \leq \gamma \leq 1. \quad (48)$$

Furthermore, if the optimal strategy is PSO, i.e. if $t_1^*(F) < t_2^*(F)$, then the first order necessary conditions (see proof of properties 3.a and 3.b) require that

$$r_{1,1}, r_{1,2} < r_{2,1}, r_{2,2}. \quad (49)$$
On the contrary, if the optimal strategy is DPR, i.e. if $t_1^*(F) > t_2^*(F)$, then the first order necessary conditions (see proof of properties 4.a and 4.b) require that

$$r_{1,1}, r_{1,2} > r_{2,1}, r_{2,2}. \quad (50)$$

Furthermore, we have for any optimal solution,

$$t_1^*(F) - t_2^*(F) = \gamma(F^{-1}(r_{1,1}) - F^{-1}(r_{2,1})) + (1 - \gamma)(F^{-1}(r_{1,2}) - F^{-1}(r_{2,2})). \quad (51)$$

As $T_1 \geq_{disp} T_2$, for any PSO optimal solution, one has

$$F_i^{-1}(r_{1,j}) - F_i^{-1}(r_{2,j}) \geq F_i^{-1}(r_{1,j}) - F_i^{-1}(r_{2,j}), \quad (52)$$

$$\text{for } j = 1, 2, \quad (53)$$

which implies that $t_1^*(F_1) - t_2^*(F_1) \geq t_1^*(F_2) - t_2^*(F_2)$. As a similar proof applies for the case of a DPR optimal solution, this amounts to $t_2^*(F_1) - t_1^*(F_1) \geq t_2^*(F_2) - t_1^*(F_2)$. ■

This proposition establishes general conditions guaranteeing that when the regulatory date process is more random, then the optimal policies are reinforced. In the case of PSO, the stockout period is increased, and in case of DPR, the dual product pipe-line inventory period is increased. This formally analyzes a conjecture, empirically given in ([4])). These authors argue that SPR is a high-risk, high-return strategy while DPR to be less risky.

6 Conclusions, managerial insights, and future research

In this paper, we apply the CVaR minimization to a product rollover problem with uncertain admissibility date. Results show that the optimal strategy depends on the cost and price parameters, on the probability distribution and the risk. We derive optimality conditions and unique closed-form solutions for SPR and DPR. Furthermore, we analyze the variation of optimal costs and solutions under different probability distribution families. Many potential extensions and directions for this research are under consideration. For instance, we are looking into an optimization with respect to a distribution free admissibility date, different products and life-cycles, and product rollover strategies for time-dependent demand. Further, we are also working on the expected value criterion under a Bass diffusion rate demand.
References


A A: Convexity Properties

Lemma 1: The loss function $L_1(\cdot, \cdot, T)$ is strictly jointly convex over $\mathbb{R}_2^2$.

Proof. We can rewrite $L_1(\cdot, \cdot, T)$ as:

$$L_1(t_1, t_2, T) = L_a(t_1, t_2, T) + L_b(t_1, t_2, T),$$

with

$$L_a(t_1, t_2, T) = c_h[T - t_2]^+ + (m_2 + g)[t_2 - T]^+$$

$$L_b(t_1, t_2, T) = -(g + m_1^1)[t_1 - T]^+ + (m_1 + g)[T - t_1]^+.$$  \hspace{1cm} (54)

Note that $L_1(\cdot, \cdot, T)$ is the non negative sum of $L_a(\cdot, \cdot, T)$ and $L_b(\cdot, \cdot, T)$, where $L_a(\cdot, \cdot, T)$ is convex and $L_b(\cdot, \cdot, T)$ is convex by assumption (8), therefore $L_1(\cdot, \cdot, T)$ is jointly convex.

Property 1: The CVaR loss function $l_{\beta,1}(t_1, t_2, \alpha)$ is strictly jointly convex.

Proof. It is a known result that if $L_1(\cdot, \cdot, T)$ is convex for any fixed value $T$, then the CvaR minimization leads to a convex problem (see Rockafellar and Uryasev, 2000, 2002). Convexity of $L_1(\cdot, \cdot, T)$ in $\mathbb{R}_1$ was previously proved in Lemma 1.

Lemma 2: The loss function $L_2(\cdot, \cdot, T)$ is strictly jointly convex over $\mathbb{R}^+ \times \mathbb{R}^+$ under the assumption $m_2 - m_1^1 - s_1 + c_h > 0$.

Proof. We can rewrite $L_2(\cdot, \cdot, T)$ as:

$$L(t_1, t_2, T) = L_c(t_1, t_2, T) + L_d(t_1, t_2, T),$$

with

$$L_c(t_1, t_2, T) = (m_2 - m_1^1 - s_1)[t_2 - T]^+ + c_h[T - t_2]^+$$

$$L_d(t_1, t_2, T) = (m_1 + g)[T - t_1]^+ + s_1[t_1 - T]^+.$$  \hspace{1cm} (57)

Note that $L_2(\cdot, \cdot, T)$ is the non negative sum of $L_c(\cdot, \cdot, T)$ and $L_d(\cdot, \cdot, T)$, with $L_d(\cdot, \cdot, T)$ convex and $L_c(\cdot, \cdot, T)$ convex if $m_2 - m_1^1 - s_1 + c_h > 0$, therefore $L_2(\cdot, \cdot, T)$ is jointly convex.

Property 2: The CVaR loss function $l_{\beta,2}(t_1, t_2, \alpha)$ is strictly jointly convex.

Proof. It is a known result that if $L_2(\cdot, \cdot, T)$ is convex for any fixed value $T$, then the CvaR minimization leads to a convex problem (see Rockafellar and Uryasev, 2000, 2002). Convexity of $L_2(\cdot, \cdot, T)$ in $\mathbb{R}_2$ was previously proved in Lemma 2.
B B-1: First Order Conditions for $l_{\beta,1}(t_1, t_2, \alpha)$

**Lemma:** The CVaR loss function $l_{\beta,1}(t_1, t_2, \alpha)$ is differentiable inside $\mathbb{R}^2$.

**Proof.** This property is direct from the expression of the derivatives of the loss function (See Appendix C).

**Property 3.a:** Consider the setting $m_2 \geq m_1 > m'_1$. Under the assumption

$$\frac{m_1 + g}{m_1 - m'_1} < \frac{C_{h,2}}{m_2 + C_{h,2} + g}, \quad (60)$$

the first-order conditions solutions

$$t^*, \text{FOC}_{\beta,1,1} = F^{-1}\left(\frac{(m_1 + g)(1 - \beta)}{m_1 - m'_1}\right), \quad (61)$$

$$t^*, \text{FOC}_{\beta,1,2} = \frac{m_1 + C_{h,2} + g}{m_2 + C_{h,2} + g} F^{-1}\left(\frac{C_{h,2} + \beta(m_2 + g)}{m_2 + C_{h,2} + g}\right) + \frac{m_2 - m_1}{m_2 + C_{h,2} + g} F^{-1}\left(\frac{C_{h,2}(1 - \beta)}{C_{h,2} + m_2 + g}\right). \quad (62)$$

corresponds to the unique finite minimum of the CVaR loss function $l_{\beta,1}(\cdot, \cdot, \cdot)$ in $\mathbb{R}_1$. Otherwise, if condition (60) is not satisfied, there is no finite minimum in $\mathbb{R}_1$ for $l_{\beta,1}(\cdot, \cdot, \cdot)$.

**Proof.** The piecewise linear function $L_1(t_1, t_2, T)$, with $t_1$, $t_2$ assumed to be given, is depicted in figure (4). On the same figure are displayed the critical values for the $\alpha$ parameters corresponding to the slope discontinuities of the $L_1(t_1, t_2, \cdot)$ function. When $m_2 \geq m_1 > m'_1$, these critical values are given, as functions of $t_1$ and $t_2$, by

$$\tilde{\alpha}_{1,1}(t_1, t_2) = m_1(t_2 - t_1) + g(t_2 - t_1), \quad (63)$$

$$\tilde{\alpha}_{1,2}(t_1, t_2) = m_2(t_2 - t_1) + g(t_2 - t_1), \quad (64)$$

$$\tilde{\alpha}_{1,3}(t_1, t_2) = m_2t_2 - m'_1t_1 + g(t_2 - t_1), \quad (65)$$

with $\tilde{\alpha}_{1,1}(t_1, t_2) \leq \tilde{\alpha}_{1,2}(t_1, t_2) \leq \tilde{\alpha}_{1,3}(t_1, t_2)$ (see Figure (4)).

In order to characterize the first order conditions, we define the following regions:

$$C_{1,1} = \{(t_1, t_2, \alpha) \mid (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,1}(t_1, t_2), \infty[, \}$$

$$C_{1,2} = \{(t_1, t_2, \alpha) \mid (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,1}(t_1, t_2), \tilde{\alpha}_{1,2}(t_1, t_2)[, \}$$

$$C_{1,3} = \{(t_1, t_2, \alpha) \mid (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,2}(t_1, t_2), \tilde{\alpha}_{1,3}(t_1, t_2)[, \}$$

$$C_{1,4} = \{(t_1, t_2, \alpha) \mid (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,3}(t_1, t_2), \infty[, \}.$$
First order conditions by regions.

**Region C\(_{1.1}\).**

In this region, the objective function becomes

\[
L_1(t_1, t_2, T) = \left[ (m_2 - m_1')t_1 + (m_1' - m_2)G(t_1) + m_1(1 - (F(t_1) + G(t_1) - t_1)) \right] + (c_{h.2} + g)(F(t_2) - F(t_1)) + g(t_2 - t_1) - \alpha \]

and the first order derivatives are given by

\[
\begin{align*}
\frac{dL_1(t_1, t_2, \alpha)}{da} &= \frac{-\beta}{1 - \beta} < 0, \\
\frac{dL_1(t_1, t_2, \alpha)}{dt_1} &= \frac{(m_1 - m_1')}{1 - \beta} F(t_1) - \frac{(m_1 + g)}{1 - \beta}, \\
\frac{dL_1(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + c_{h.2} + g)}{1 - \beta} F(t_2) - \frac{c_{h.2}}{1 - \beta}.
\end{align*}
\]

**Region C\(_{1.2}\).**

According to Figure 4, let’s define \(T_2(\alpha, t_1, t_2)\) as the \(T\) value solving

\[
\alpha = m_1(T - t_1) + g(t_2 - t_1) + m_2(t_2 - T)
\]

and \(T_3(\alpha, t_1, t_2)\) as the \(T\) value solving

\[
\alpha = m_1(T - t_1) + g(t_2 - t_1) + (c_{h.2} + g)(T - t_2).
\]
In this region, the objective function becomes
\[ l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left( (m_1 - m_1') t_1 F(t_1) + (m_1' - m_1) G(t_1) + (m_1 - m_2) G(T_2(t_1, t_2, \alpha)) + (-m_1 t_1 + g(t_2 - t_1) + m_2 t_2 - \alpha) F(T_2(t_1, t_2, \alpha)) + (m_1 + c_{h,2} + g) (\mu - G(T_3(t_1, t_2, \alpha))) + (-m_1 t_1 - g t_1 - c_{h} t_2 - \alpha) (1 - F(T_3(t_1, t_2, \alpha))) \right) \] (72)

and the first order derivatives are given by
\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_1 - m_1')}{1 - \beta} F(t_1) - \frac{(m_1 + g)}{1 - \beta} F(T_2(\alpha, t_1, t_2)) \] (73)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} F(T_2(t_1, t_2, \alpha)) - \frac{c_{h,2}}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2, \alpha))) \] (74)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(t_1, t_2, \alpha)) - F(T_2(t_1, t_2, \alpha)) - \beta}{1 - \beta} \] (75)

**Region C_{1,3}**.
According to Figure 4, let’s define \( T_1(\alpha, t_1, t_2) \) as the \( T \) value solving
\[ \alpha = -m_1'(t_1 - T) + g(t_2 - t_1) + m_2(t_2 - T). \] (76)

In this region, the objective function becomes
\[ l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left( (m_1 - m_2) G(T_1(t_1, t_2, \alpha)) + (m_1 + c_{h,2} + g) (\mu - G(T_3(t_1, t_2, \alpha))) + (-m_1 t_1 + g(t_2 - t_1) + m_2 t_2 - \alpha) F(T_1(t_1, t_2, \alpha)) + (-m_1 t_1 - g t_1 - c_{h} t_2 - \alpha) (1 - F(T_3(t_1, t_2, \alpha))) \right) \] (77)

and the first order derivatives are given by
\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1' + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) - \frac{(m_1 + g)}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))) \] (78)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) - \frac{c_{h,2}}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))) \] (79)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha)) - \beta}{1 - \beta} \] (80)

**Region C_{1,4}**.
In this region, the objective function becomes
\[ l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_1 + c_{h,2} + g) (\mu - G(T_3(t_1, t_2, \alpha))) + (-m_1 t_1 - g t_1 - c_{h} t_2 - \alpha) (1 - F(T_3(t_1, t_2, \alpha))) \right] \] (81)

The first order derivatives are given by:
\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1 + g)}{1 - \beta} \left( 1 - F(T_3(t_1, t_2, \alpha)) \right) \] (82)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = -\frac{c_{h,2}}{1 - \beta} \left( 1 - F(T_3(t_1, t_2, \alpha)) \right) \] (83)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(t_1, t_2, \alpha)) - \beta}{1 - \beta} \] (84)

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**Analysis of the first order conditions.**

Direct computations show that the only case where the first order conditions have a solution is the region $C_{1.2}$. Under adequate parameters assumptions, the first order conditions associated to (73)-(75) have the solution

$$
t_{\beta,1.1}^{*,\text{FOC}} = F^{-1}\left(\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'}\right),
$$

$$
t_{\beta,1.2}^{*,\text{FOC}} = F^{-1}\left(\frac{m_1 + c_{h,2} + g}{m_1 - m_1'}\left(\frac{c_{h,2}(1 - \beta)}{m_2 + c_{h,2} + g}\right)\right),
$$

$$
t_{\beta,1.2}^{*,\text{FOC}} = F^{-1}\left(\frac{m_1 + c_{h,2} + g}{m_1 - m_1'}\left(\frac{c_{h,2}(1 - \beta)}{m_2 + c_{h,2} + g}\right)\right) - \frac{(m_1 + g)t_{\beta,1.1}^{*,\text{FOC}} - c_{h,2}t_{\beta,1.2}^{*,\text{FOC}}}{m_2 + c_{h,2} + g},
$$

and we also have the following parameter values

$$
T_3(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}, a_{\beta,1}^{*,\text{FOC}}) = F^{-1}\left(\frac{c_{h,2}(m_2 + g)}{m_2 + c_{h,2} + g}\right),
$$

$$
T_2(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}, a_{\beta,1}^{*,\text{FOC}}) = F^{-1}\left(\frac{c_{h,2}(1 - \beta)}{m_2 + c_{h,2} + g}\right).
$$

In order to guarantee that solution (85)-(87) exists and belongs to the interior of $C_{1.2}$, the following assumptions are required

$$
\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'} < 1,
$$

$$
t_{\beta,1.1}^{*,\text{FOC}} < t_{\beta,1.2}^{*,\text{FOC}},
$$

$$
\tilde{a}_{1.1}(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}) < a_{\beta,1}^{*,\text{FOC}} < a_{1.2}(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}).
$$

From Figure (4) it can be seen that condition (92) is equivalent to

$$
t_{\beta,1.1}^{*,\text{FOC}} \leq T_1(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}, a_{\beta,1}^{*,\text{FOC}}),
$$

$$
T_2(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}, a_{\beta,1}^{*,\text{FOC}}) \leq t_{\beta,1.2}^{*,\text{FOC}},
$$

$$
t_{\beta,1.2}^{*,\text{FOC}} \leq T_3(t_{\beta,1.1}^{*,\text{FOC}}, t_{\beta,1.2}^{*,\text{FOC}}, a_{\beta,1}^{*,\text{FOC}}),
$$

and it is direct to see that conditions (93)-95 amount to the parameters condition

$$
\frac{m_1 + g}{m_1 - m_1'} < \frac{c_{h,2}}{m_2 + c_{h,2} + g},
$$

which furthermore guarantees (90)-(91).

**Property 3.b:** Consider the setting $m_1 > m_2 \geq m_1'$. Under assumptions

$$
m_1' < -g
$$

$$
m_2 > m_1 - c_{h,2}
$$

$$
m_1' > m_1 - c_{h,2}
$$

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and for
\[ \beta \in [\beta_{1,inf}, 1 + \frac{m_2 - m_1}{c_{h,2}}], \]
with
\[ \beta_{1,inf} = \begin{cases} 0 & \text{if } m_1 - m_1' > m_2 + g + c_{h,2}, \\ \beta_{1,inf} & \text{if } m_1 - m_1' \leq m_2 + g + c_{h,2}, \end{cases} \]

the first-order conditions solutions
\[ t_{*, FOC}^{\beta,1} = F^{-1}\left(\frac{m_2 - m_2'}{m_1 - m_1'}\right) + F^{-1}\left(\frac{m_2 - m_2'}{m_1 - m_1'} \cdot \frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'}\right), \]
\[ t_{*, FOC}^{\beta,2} = F^{-1}\left(\frac{m_1 + g + c_{h,2}\beta}{m_2 + g + c_{h,2}}\right). \]

This corresponds to the unique finite minimum of the CVaR loss function \( l_{3,1}(\cdot, \cdot) \) in \( R_1 \). While if condition (97),(98) or (100) is not satisfied, there is no finite minimum in \( R_1 \).

**Proof.** When, \( m_1 > m_2 \geq m_1' \), it can be seen that the critical values for the \( \alpha \) parameters corresponding to the slope discontinuities for the piecewise linear function \( L_1(t_1, t_2, \cdot) \), as functions of \( t_1 \) and \( t_2 \), are given by
\[ \hat{\alpha}_{1,1}(t_1, t_2) = m_2(t_2 - t_1) + g(t_2 - t_1), \]
\[ \hat{\alpha}_{1,2}(t_1, t_2) = m_1(t_2 - t_1) + g(t_2 - t_1), \]
\[ \hat{\alpha}_{1,3}(t_1, t_2) = m_2 t_2 - m_1 t_1 + g(t_2 - t_1). \]

The ordering of these critical values conditions the structure of the optimality conditions. As this ordering depends on the combination of the parameters and of the \( t_1 \) and \( t_2 \) values, two cases have to be considered: \( \hat{\alpha}_{1,1}(t_1, t_2) \leq \hat{\alpha}_{1,2}(t_1, t_2) \leq \hat{\alpha}_{1,3}(t_1, t_2) \), defined as Case A (see Figure (5)) and \( \hat{\alpha}_{1,1}(t_1, t_2) \leq \hat{\alpha}_{1,3}(t_1, t_2) \leq \hat{\alpha}_{1,2}(t_1, t_2) \), defined as Case B, (see Figure (6)).
Figure 5: Function $L_1(t_1, t_2, T)$: CASE A

Figure 6: Function $L_1(t_1, t_2, T)$: CASE B
Case A.

In order to characterize the first order conditions, we define the regions for the case

\[ \tilde{\alpha}_{1,1}(t_1, t_2) \leq \alpha(t_1, t_2) \leq \tilde{\alpha}_{1,3}(t_1, t_2) \]  

as

\[
\begin{align*}
C_{1,1} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_1 \text{ and } \alpha \in [\tilde{\alpha}_{1,1}(t_1, t_2), \tilde{\alpha}_{1,3}(t_1, t_2)] \}, \\
C_{1,2} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_1 \text{ and } \alpha \in [\alpha_{1,1}(t_1, t_2), \tilde{\alpha}_{1,2}(t_1, t_2)] \}, \\
C_{1,3} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_1 \text{ and } \alpha \in [\tilde{\alpha}_{1,2}(t_1, t_2), \tilde{\alpha}_{1,3}(t_1, t_2)] \}, \\
C_{1,4} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_1 \text{ and } \alpha \in [\tilde{\alpha}_{1,3}(t_1, t_2), \infty] \}.
\end{align*}
\]

**First order conditions by region.**

**Region C_{1,1}**.

In this region, the objective function becomes

\[
\begin{align*}
l_{\beta,1}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1 - \beta} \left( (m_2 t_2 - m_1 t_1) F(t_1) + (m_1 - m_2) G(t_1) \right. \\
&\quad + \left. m_1 (\mu - G(t_1) - t_1 (1 - F(t_1))) + m_2 t_2 (F(t_2) - F(t_1)) \right. \\
&\quad + \left. (c_{\theta,2} + g)(\mu - G(t_2) - t_2 (1 - F(t_2))) + g(t_2 - t_1) - \alpha \right)
\end{align*}
\]

and the first order derivatives are given by

\[
\begin{align*}
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} &= \frac{-\beta}{1 - \beta} < 0, \\
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} &= \frac{(m_1 - m_2)}{1 - \beta} F(t_1) - \frac{(m_1 + g)}{1 - \beta}, \\
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + c_{\theta,2} + g)}{1 - \beta} F(t_2) - \frac{c_{\theta,2}}{1 - \beta}.
\end{align*}
\]

**Region C_{1,2}**.

According to Figure 5, let’s define \(T_1(\alpha, t_1, t_2)\) as the \(T\) value solving

\[ \alpha = -m_1'(t_2 - T) + g(t_2 - t_1) + m_2 t_2 - T. \]  

and \(T_2(\alpha, t_1, t_2)\) as the \(T\) value solving

\[ \alpha = m_1(T - t_1) + g(t_2 - t_1) + m_2 t_2 - T. \]

In this region, the objective function becomes

\[
\begin{align*}
l_{\beta,1}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1 - \beta} \left( -(m_1' + g) t_1 + (m_2 + g) t_2 - \alpha \right) F(T_1(t_1, t_2, \alpha)) \\
&\quad + (m_1' - m_2) G(T_1(t_1, t_2, \alpha)) \\
&\quad + (m_1 - m_2)(G(t_2) - G(T_2(t_1, t_2, \alpha))) \\
&\quad + (-m_1 t_1 + g(t_2 - t_1) + m_2 t_2 - \alpha)(F(t_2) - F(T_2(t_1, t_2, \alpha))) \\
&\quad + (m_1 + c_{\theta,2} + g)(\mu - G(t_2)) \\
&\quad + (-m_1 t_1 - g(t_1 - c_{\theta,2} - \alpha)(1 - F(t_2)))
\end{align*}
\]

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and the first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1 + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) - \frac{(m_1 + g)(1 - F(T_2(\alpha, t_1, t_2)))}{1 - \beta},
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} (F(T_1(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2))) - \frac{c_{h,2}}{1 - \beta} (1 - F(t_2)) + \frac{(m_1 + g)}{1 - \beta} F(t_2),
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_2(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha))}{1 - \beta} - \beta,
\]

Region C_{1.3}.

According to Figure 5, let’s define \( T_3(\alpha, t_1, t_2) \) as the \( T \) value solving

\[
\alpha = m_1(T - t_1) + g(t_2 - t_1) + (c_{h,2} + g)(T - t_2)
\]

In this region, the objective function becomes

\[
l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_1 - m_2)G(T_1(t_1, t_2, \alpha)) + (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1, t_2, \alpha))) + (-m_1' t_1 + g(t_2 - t_1) + m_2 t_2 - \alpha)F(T_1(t_1, t_2, \alpha)) + (-m_1 t_1 - g t_1 - c_{h,2} t_2 - \alpha)(1 - F(T_3(t_1, t_2, \alpha))) \right]
\]

and the first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1 + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) - \frac{(m_1 + g)(1 - F(T_3(\alpha, t_1, t_2)))}{1 - \beta},
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} (F(T_1(\alpha, t_1, t_2)) - F(T_3(\alpha, t_1, t_2))) - \frac{c_{h,2}}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))),
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha))}{1 - \beta} - \beta.
\]

Region C_{1.4}.

In this region, the objective function becomes

\[
l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1, t_2, \alpha))) + (-m_1 t_1 - g t_1 - c_{h,2} t_2 - \alpha)(1 - F(T_3(t_1, t_2, \alpha))) \right]
\]

and the first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1 + g)}{1 - \beta} (1 - F(T_3(t_1, t_2, \alpha))),
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = -\frac{c_{h,2}}{1 - \beta} (1 - F(T_3(t_1, t_2, \alpha))),
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(t_1, t_2, \alpha)) - \beta}{1 - \beta}.
\]
Analysis of the first order conditions. It is direct to see that the only case where the first order conditions possibly have a solution is the region \( C_{1,2} \). Under adequate assumptions, the first order conditions associated to (115)-(117) have the solution

\[
\begin{align*}
t_{\beta,1,1}^{*,\text{FOC}} &= \frac{(m_2 - m_1')}{m_1 - m_1'} f^{-1}\left(\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'}\right), \\
t_{\beta,1,2}^{*,\text{FOC}} &= f^{-1}\left(\frac{m_1 + g + c_h 2\beta}{m_2 + g + c_h}\right), \\
\alpha_{\beta,1}^{*,\text{FOC}} &= \left(\frac{m_1 + c_h 2\beta}{m_1 + c_h}\right)^{-1}\left(\frac{(m_1 - \beta m_1' + g(1 - \beta))}{m_1 - m_1'} - (m_1 + g) \alpha_{\beta,1}^{*,\text{FOC}} - c_h \alpha_{\beta,1}^{*,\text{FOC}}\right)
\end{align*}
\]

and we also find the following parameter values

\[
\begin{align*}
\alpha_{\beta,1}^{*,\text{FOC}} &= \left(\frac{m_1 + c_h 2\beta}{m_1 + c_h}\right)^{-1}\left(\frac{(m_1 - \beta m_1' + g(1 - \beta))}{m_1 - m_1'} - (m_1 + g) \alpha_{\beta,1}^{*,\text{FOC}} - c_h \alpha_{\beta,1}^{*,\text{FOC}}\right)
\end{align*}
\]

In order to guarantee that this solution belongs to the interior of \( C_{1,2} \), the following assumptions are required

\[
\begin{align*}
\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'} &< 1, \\
\frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'} &< 1, \\
\frac{m_1 + c_h 2\beta}{m_2 + g + c_h}\left(\frac{t_{\beta,1,1}^{*,\text{FOC}}}{t_{\beta,1,2}^{*,\text{FOC}}} - \frac{\alpha_{\beta,1}^{*,\text{FOC}}}{\alpha_{\beta,1}^{*,\text{FOC}}}\right) &< 1.
\end{align*}
\]

From Figure (5) it can be seen that, under assumption (135), condition (136) is equivalent to

\[
\begin{align*}
0 &\leq T_1(t_{\beta,1,1}^{*,\text{FOC}}, t_{\beta,1,2}^{*,\text{FOC}}, \alpha_{\beta,1}^{*,\text{FOC}}) < t_{\beta,1,1}^{*,\text{FOC}}, \\
t_{\beta,1,1}^{*,\text{FOC}} &\leq T_2(t_{\beta,1,1}^{*,\text{FOC}}, t_{\beta,1,2}^{*,\text{FOC}}, \alpha_{\beta,1}^{*,\text{FOC}}) < t_{\beta,1,2}^{*,\text{FOC}},
\end{align*}
\]

which amounts to

\[
\frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'} \leq \frac{m_1 + g + c_h 2\beta}{m_2 + g + c_h}.
\]

The linear \( \beta \) functions \( \frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'} \) converges to \( \frac{m_1 + g}{m_1 - m_1'} \) when \( \beta \) converges to 0 and to 1 when \( \beta \) converges to 1, while the function \( \frac{m_1 + g + c_h 2\beta}{m_2 + g + c_h} \) respectively converges to \( \frac{m_1 + g}{m_2 + g + c_h} \) and \( \frac{m_1 + g + c_h 2\beta}{m_2 + g + c_h} > 1 \). It is easily seen that if

\[
m_1 - m_1' \geq m_2 + g + c_h
\]

condition (140) is satisfied for any \( \beta \) in the interval \([0, 1]\), while if

\[
m_1 - m_1' < m_2 + g + c_h
\]

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condition (140) is satisfied for any \( \beta \) satisfying

\[
\beta \geq \frac{(m_1 + g)(m_2 + g + c_{h,2} - m_1 + m_1')}{(m_1' + g)(m_2 + g) + c_{h,2}(m_1 + g)}.
\]  

(143)

It is worth noting that under conditions \( m_1 > m_2 \geq m_1' \), \( m_1' < -g \), the term (143) is greater than 0.

**Case B**

In order to characterize the first order conditions, we define the regions for the case

\[
\tilde{\alpha}_{1,1}(t_1, t_2) \leq \tilde{\alpha}_{1,3}(t_1, t_2) \leq \tilde{\alpha}_{1,2}(t_1, t_2)
\]

(144)

In order to characterize the first order conditions, we define the regions

\[
\begin{align*}
C_{1,1} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\infty, \tilde{\alpha}_{1,1}(t_1, t_2)[, \\
C_{1,2} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,1}(t_1, t_2), \tilde{\alpha}_{1,3}(t_1, t_2)[, \\
C_{1,3} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,3}(t_1, t_2), \tilde{\alpha}_{1,2}(t_1, t_2)[, \\
C_{1,4} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_1 \text{ and } \alpha \in ]\tilde{\alpha}_{1,2}(t_1, t_2), \infty[.
\end{align*}
\]

**First order conditions by region.**

**Region C_{1,1}.
** In this region, the objective function is

\[
l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[(m_2 t_2 - m_1' t_1) F(t_1) + (m_1' - m_2) G(t_1) + m_1 (\mu - G(t_1) - t_1 (1 - F(t_1))) + m_2 t_2 (F(t_2) - F(t_1)) + (c_{h,2} + g)(\mu - G(t_2) - t_2 (1 - F(t_2))) + g(t_2 - t_1) - \alpha \right]
\]

(145)

and the first order derivatives are given by

\[
\begin{align*}
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} &= \frac{-\beta}{1 - \beta} < 0, \\
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} &= \frac{(m_1 - m_1')}{1 - \beta} F(t_1) - \frac{(m_1 + g)}{1 - \beta}, \\
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + c_{h,2} + g)}{1 - \beta} F(t_2) - \frac{c_{h,2}}{1 - \beta}.
\end{align*}
\]

(146) \hspace{1cm} (147) \hspace{1cm} (148)

**Region C_{1,2}.
** According to Figure 6, let’s define \( T_1(\alpha, t_1, t_2) \) as the \( T \) value solving

\[
\alpha = -m_1' (t_1 - T) + g(t_2 - t_1) + m_2 (t_2 - T).
\]

(149)

and \( T_2(\alpha, t_1, t_2) \) as the \( T \) value solving

\[
\alpha = m_1(T - t_1) + g(t_2 - t_1) + m_2 (t_2 - T)
\]

(150)
The objective function is given by

\[ l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left( -(m_1' + g)t_1 + (m_2 + g)t_2 - \alpha \right) F(T_1(t_1, t_2, \alpha)) + (m_1' - m_2)G(T_1(t_1, t_2, \alpha)) + (m_1 - m_2)(G(t_2) - G(T_2(t_1, t_2, \alpha))) + (-m_1t_1 + g(t_2 - t_1) + m_2t_2 - \alpha)(F(t_2) - F(T_2(t_1, t_2, \alpha))) + (m_1 + c_{h,2} + g)(\mu - G(t_2)) + (-m_1t_1 - gt_1 - c_{h,2}t_2 - \alpha)(1 - F(t_2)) \] (151)

and the first order derivatives are given by:

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1' + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) + \frac{(m_1 + g)}{1 - \beta} F(T_2(\alpha, t_1, t_2)) - \frac{(m_1 + g)}{1 - \beta}, \] (152)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} (F(T_1(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2))) - \frac{c_{h,2}}{1 - \beta} (1 - F(t_2)) + \frac{(m_1 + g)}{1 - \beta} F(t_2), \] (153)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_2(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha))}{1 - \beta}. \] (154)

**Region C_{1,3}.**

According to Figure 6, let's define \( T_3(\alpha, t_1, t_2) \) as the \( T \) value solving

\[ \alpha = m_1(T - t_1) + g(t_2 - t_1) + (c_{h,2} + g)(T - t_2) \] (155)

The objective function is given by

\[ l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_1 - m_2)(G(t_2) - G(T_2(t_1, t_2, \alpha))) + (m_1 + c_{h,2} + g)(\mu - G(t_2)) + (-m_1t_1 + g(t_2 - t_1) + m_2t_2 - \alpha)(F(t_2) - F(T_2(t_1, t_2, \alpha))) + (-m_1t_1 + gt_1 - c_{h,2}t_2 - \alpha)(1 - F(t_2)) \right]. \] (156)

and the first order derivatives are given by:

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1 + g)}{1 - \beta} (F(t_2) - F(T_2(\alpha, t_1, t_2))) - \frac{(m_1 + g)}{1 - \beta} (1 - F(t_2)), \] (157)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} (F(t_2) - F(T_2(\alpha, t_1, t_2))) - \frac{c_{h,2}}{1 - \beta} (1 - F(t_2)), \] (158)

\[ \frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_2(t_1, t_2, \alpha))}{1 - \beta}. \] (159)

**Region C_{1,4}.**

The objective function is given by

\[ l_{\beta,1}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_1 + c_{h,2} + g)(\mu - G(T_3(t_1, t_2, \alpha))) + (-m_1t_1 + gt_1 - c_{h,2}t_2 - \alpha)(1 - F(T_3(t_1, t_2, \alpha))) \right]. \] (160)
and the first order derivatives are given by:

\[
\frac{dl_{t_1,1}(t_1, t_2, a)}{dt_1} = \left(\frac{m_1 + g}{m_1 - m_1'}\right) \left(1 - F(T_3(t_1, t_2, a))\right) < 0,
\]

\[
\frac{dl_{t_1,1}(t_1, t_2, a)}{dt_2} = \left(\frac{e_{h,2}}{1 - \beta}\right) \left(1 - F(T_3(t_1, t_2, a))\right) < 0,
\]

\[
\frac{dl_{t_1,1}(t_1, t_2, a)}{d\alpha} = \frac{F(T_3(t_1, t_2, a)) - \beta}{1 - \beta}.
\]

**Analysis of the first order conditions.** It is direct to see that the only case where the first order conditions possibly have a solution is the region \(C_{1,2}\). Under adequate assumptions, the first order conditions (152)-(154) have the solution

\[
t^{\ast,FOC}_{\beta,1,1} = \left(\frac{m_2 - m_1'}{m_1 - m_1'}\right) F^{-1}\left(\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'}\right) + \left(\frac{m_1 - m_2}{m_1 - m_1'}\right) F^{-1}\left(\frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'}\right),
\]

\[
t^{\ast,FOC}_{\beta,1,2} = F^{-1}\left(\frac{m_1 + g + e_{h,2}\beta}{m_2 + g + e_{h,2}}\right),
\]

\[
a^{\ast,FOC} = \left(\frac{m_1 + e_{h,2} + g}{m_1 - m_1'}\right) F^{-1}\left(\frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'}\right) - \left(\frac{m_1 + g}{m_1 - m_1'}\right) t^{\ast,FOC}_{\beta,1,1} - c_{\beta,1,1} t^{\ast,FOC}_{\beta,1,2}.
\]

We also find the following parameter values

\[
T_1(t^{\ast,FOC}_{\beta,1,1}, t^{\ast,FOC}_{\beta,1,2}, a^{\ast,FOC}) = F^{-1}\left(\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'}\right),
\]

\[
T_2(t^{\ast,FOC}_{\beta,1,1}, t^{\ast,FOC}_{\beta,1,2}, a^{\ast,FOC}) = F^{-1}\left(\frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'}\right).
\]

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of \(C_{1,2}\). Basically these assumptions are the following

\[
\frac{(m_1 + g)(1 - \beta)}{m_1 - m_1'} < 1,
\]

\[
\frac{m_1 - \beta m_1' + g(1 - \beta)}{m_1 - m_1'} < 1,
\]

\[
\frac{m_1 + g + e_{h,2}\beta}{m_2 + g + e_{h,2}} < 1,
\]

\[
t^{\ast,FOC}_{\beta,1,1} < t^{\ast,FOC}_{\beta,1,2},
\]

\[
a^{\ast,FOC} < a_{\beta,1,1}^{\ast,FOC} < a_{\beta,1,2}^{\ast,FOC}.
\]

From Figure (6) it can be seen that condition (173) is equivalent to

\[
0 \leq T_1(t^{\ast,FOC}_{\beta,1,1}, t^{\ast,FOC}_{\beta,1,2}, a^{\ast,FOC}) \leq t^{\ast,FOC}_{\beta,1,1},
\]

\[
t^{\ast,FOC}_{\beta,1,1} \leq T_2(t^{\ast,FOC}_{\beta,1,1}, t^{\ast,FOC}_{\beta,1,2}, a^{\ast,FOC}) \leq t^{\ast,FOC}_{\beta,1,2},
\]

\[
t^{\ast,FOC}_{\beta,1,1} \leq T_2(t^{\ast,FOC}_{\beta,1,1}, t^{\ast,FOC}_{\beta,1,2}, a^{\ast,FOC}) \leq t^{\ast,FOC}_{\beta,1,2}.
\]
which amounts to
\[
\frac{m_1 - \beta m'_1 + g(1 - \beta)}{m_1 - m'_1} \leq \frac{m_1 + g + c_{h,2} \beta}{m_2 + g + c_{h,2}}.
\] (176)

The Case A analysis also applies and gives the final condition (100).

C Appendix B-2: First Order Conditions for \( l_{\beta,2}(t_1, t_2, \alpha) \)

**Lemma:** The CVaR loss function \( l_{\beta,2}(t_1, t_2, \alpha) \) is differentiable inside \( R^3_+ \).

**Proof.** This property is direct from the expression of the derivatives of the loss function (See Appendix C).

**Property 4.a:** Consider the setting \( c_{h,2} > s_1 \). Under the assumption
\[
\frac{m_2 - m'_1 - s_1}{m_1 + g} \geq 0, \quad \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}, \quad (177)
\]
\[
\frac{c_{h,2}}{m_1 + g + s_1} > \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}, \quad (178)
\]

the first-order conditions solutions
\[
l^{*,FOC}_{\beta,2,1} = F^{-1}\left(\frac{m_1 + g + s_1 \beta}{m_1 + g + s_1}\right), \quad (179)
\]
\[
l^{*,FOC}_{\beta,2,2} = \left(\frac{m_2 - m'_1}{m_2 - m'_1 - s_1 + c_{h,2}}\right) F^{-1}\left(\frac{c_{h,2}(1 - \beta)}{m_2 - m'_1 - s_1 + c_{h,2}}\right)
+ \left(\frac{c_{h,2} - s_1}{m_2 - m'_1 - s_1 + c_{h,2}}\right) F^{-1}\left(\frac{c_{h,2} + \beta(m_2 - m'_1 - s_1)}{m_2 - m'_1 - s_1 + c_{h,2}}\right). \quad (180)
\]

If \( t^{*,FOC}_{\beta,1,2} \geq t^{*,FOC}_{\beta,2,2} \), then \( (t^{*,FOC}_{\beta,1,2}, t^{*,FOC}_{\beta,2,2}) \) corresponds to the unique finite minimum of the CVaR loss function \( l_{\beta,2}(\cdot, \cdot, \cdot) \) in \( R_2 \). While if condition (177), or (178) is not satisfied, there is no finite minimum in \( R_2 \) for \( l_{\beta,2}(\cdot, \cdot, \cdot) \).

**Proof.** According to equation (4), it can be seen that the critical values for the \( \alpha \) parameters corresponding to the slope discontinuities for the piecewise linear function...
$L_2(t_1, t_2, \cdot)$, as functions of $t_1$ and $t_2$, are given by

$$\tilde{\omega}_{2,1}(t_1, t_2) = s_1(t_1 - t_2), \quad (181)$$
$$\tilde{\omega}_{2,2}(t_1, t_2) = (m_2 - m'_1)t_2 + s_1(t_1 - t_2), \quad (182)$$
$$\tilde{\omega}_{2,3}(t_1, t_2) = c_3(t_1 - t_2). \quad (183)$$

The ordering of these critical values conditions the structure of the optimality conditions. As this ordering depends on the combination of the parameters and of the $t_1$ and $t_2$ values, two cases have to be considered: $\tilde{\omega}_{2,1}(t_1, t_2) \leq \tilde{\omega}_{2,2}(t_1, t_2) \leq \tilde{\omega}_{2,3}(t_1, t_2)$, defined as Case A (see Figure (7)) and $\tilde{\omega}_{2,1}(t_1, t_2) \leq \tilde{\omega}_{2,3}(t_1, t_2) \leq \tilde{\omega}_{2,2}(t_1, t_2)$, defined as Case B, (see Figure (6)).

**Case A**

In this situation, the following inequalities hold (see Figure (8)): $\tilde{\omega}_{2,1}(t_1, t_2) \leq \tilde{\omega}_{2,2}(t_1, t_2) \leq \tilde{\omega}_{2,3}(t_1, t_2)$,

![Figure 7: Function $L_2(t_1, t_2, T)$: CASE A](image)
According to Figure 7, let’s define $C_{2,3}$ and $C_{2,4}$, as

\[
\begin{align*}
C_{2,1} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_2 \text{ and } \alpha \in ]\infty, \alpha_{2,1}(t_1, t_2)[, \]
\end{align*}
\]

\[
\begin{align*}
C_{2,2} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_2 \text{ and } \alpha \in ]\alpha_{2,1}(t_1, t_2), \alpha_{2,2}(t_1, t_2)[, \]
\end{align*}
\]

\[
\begin{align*}
C_{2,3} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_2 \text{ and } \alpha \in ]\alpha_{2,2}(t_1, t_2), \alpha_{2,3}(t_1, t_2)[, \]
\end{align*}
\]

\[
\begin{align*}
C_{2,4} &= \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}_2 \text{ and } \alpha \in ]\alpha_{2,3}(t_1, t_2), \infty[, \}
\end{align*}
\]

**First order conditions by region.**

**Region $C_{2,1}$.** In this region, the objective function is

\[
\begin{align*}
l_{\beta,2}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1 - \beta} \left[ ((m_2 - m_1')t_2 + s_1(t_1 - t_2))F(t_2) + (m_1 - m_2)G(t_2) \\
&\quad + (s_1t_1 - c_h t_2)(F(t_1) - F(t_2)) + (c_{h,2} - s_1)(G(t_1) - G(t_2)) \\
&\quad - ((m_1 + g)t_1 + c_h t_2)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) - \alpha \right] \quad (184)
\end{align*}
\]

and the first order derivatives are given by

\[
\begin{align*}
\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} &= \frac{-\beta}{1 - \beta} < 0, \quad (185) \\
\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} &= (s_1 + m_1 + g)F(t_1) - (m_1 + g), \quad (186) \\
\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} &= (m_2 - m_1' - s_1 + c_{h,2})F(t_2) - c_{h,2}, \quad (187)
\end{align*}
\]

**Region $C_{2,2}$.**

According to Figure 7, let’s define $T_1(\alpha, t_1, t_2)$ as the $T$ value solving

\[
\begin{align*}
\alpha &= -m_1'(t_2 - T) + m_2(t_2 - T) + s_1(t_1 - t_2) \quad (188)
\end{align*}
\]

and $T_2(\alpha, t_1, t_2)$ as the $T$ value solving

\[
\begin{align*}
\alpha &= s_1(t_1 - T) + c_h(T - t_2) \quad (189)
\end{align*}
\]

In this region, the objective function is

\[
\begin{align*}
l_{\beta,2}(t_1, t_2, \alpha) &= \alpha + \frac{1}{1 - \beta} \left[ (m_2 - m_1')t_2 + s_1(t_1 - t_2) - \alpha \right]F(T_1(\alpha, t_1, t_2)) \\
&\quad + \frac{1}{1 - \beta} [m_1 - m_2]G(T_1(\alpha, t_1, t_2)) \\
&\quad + \frac{1}{1 - \beta} \left[ (s_1t_1 - c_h t_2 - \alpha)(F(t_1) - F(T_2(\alpha, t_1, t_2))) \\
&\quad + (c_{h,2} - s_1)(G(t_1) - G(T_2(\alpha, t_1, t_2))) \right] \\
&\quad + \frac{1}{1 - \beta} \left[ -(m_1 + g)t_1 + c_h t_2 + \alpha)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) \right] \quad (190)
\end{align*}
\]
According to Figure 7, let’s define $T_3(\alpha, t_1, t_2)$ as the $T$ value solving

$$\alpha = m_1(T - t_1) + g(T - t_1) + c_h(T - t_2). \quad (194)$$

In this region, the objective function is

$$l_{s,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (s_1 t_1 - c_h t_2 - \alpha)(F(t_1) - F(T_2(\alpha, t_1, t_2))) + (c_h - s_1)(G(t_1) - G(T_2(\alpha, t_1, t_2))) + \frac{1}{1 - \beta} \left[ ((m_1 + g)t_1 + c_h t_2 + \alpha)(1 - F(t_1)) + (m_1 + c_h g)(G(t_1)) \right] \right] \quad (195)$$

and the first order derivatives are given by:

$$\frac{dl_{s,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1}{1 - \beta} (F(t_1) - F(T_2(t_1, t_2, \alpha))) - \frac{1}{1 - \beta} (m_1 + g)(1 - F(t_1)), \quad (196)$$

$$\frac{dl_{s,2}(t_1, t_2, \alpha)}{dt_2} = \frac{c_h}{1 - \beta} (1 - F(T_2(t_1, t_2, \alpha))), \quad (197)$$

$$\frac{dl_{s,2}(t_1, t_2, \alpha)}{d\alpha} = F(T_2(t_1, t_2, \alpha)) - \frac{\beta}{1 - \beta}. \quad (198)$$

**Region C_{2.3}**.

In this region, the objective function is

$$l_{s,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ ((m_1 + g)t_1 + c_h t_2 + \alpha)(1 - F(T_3(t_1, t_2, \alpha))) + (m_1 + c_h g)(\mu - G(T_3(t_1, t_2, \alpha))) \right], \quad (199)$$

and the first order derivatives are given by:

$$\frac{dl_{s,2}(t_1, t_2, \alpha)}{dt_1} = \frac{-1}{1 - \beta} (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2))), \quad (200)$$

$$\frac{dl_{s,2}(t_1, t_2, \alpha)}{dt_2} = \frac{c_h}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))), \quad (201)$$

$$\frac{dl_{s,2}(t_1, t_2, \alpha)}{d\alpha} = F(T_3(\alpha, t_1, t_2)) - \frac{\beta}{1 - \beta}. \quad (202)$$

**Analysis of the first order conditions.** It is direct to see that the only case where the first order conditions possibly have a solution is the region $C_{2.2}$. Under adequate assumptions, the first order conditions (191)-(193) have the solution

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Several assumptions are required in order to guarantee that this solution belongs to the interior of $C_{2,2}$:

\[
\frac{c_{h,2} + \beta(m_2 - m'_1 - s_1)}{m_2 - m'_1 - s_1 + c_{h,2}} < 1, \quad (208)
\]

\[
\frac{c_{h}(1 - \beta)}{m_2 - m'_1 - s_1 + c_{h,2}} < 1, \quad (209)
\]

\[
\frac{c_{h,2} + \beta(m_2 - m'_1 - s_1)}{m_2 - m'_1 - s_1 + c_{h,2}} < t_{\beta,2,2}^{*, FOC}, \quad (210)
\]

\[
\alpha_{2,1}(t_{\beta,2,1}^{*, FOC}, t_{\beta,2,2}^{*, FOC}) < \alpha^{*} < \alpha_{2,2}(t_{\beta,2,1}^{*, FOC}, t_{\beta,2,2}^{*, FOC}). \quad (211)
\]

From Figure (7) it can be seen that the last condition is equivalent to

\[
0 \leq T_1(t_{\beta,2,1}^{*, FOC}, t_{\beta,2,2}^{*, FOC}, \alpha_{2,2}^{*, FOC}) \leq t_{\beta,2,2}^{*, FOC}, \quad (212)
\]

\[
t_{\beta,2,2}^{*, FOC} \leq T_2(t_{\beta,2,1}^{*, FOC}, t_{\beta,2,2}^{*, FOC}, \alpha_{2,2}^{*, FOC}) \leq t_{\beta,2,1}^{*, FOC}. \quad (213)
\]

Condition (208) is equivalent to $m_2 - m'_1 - s_1 \geq 0$. Conditions (212)-(213) thus amount to

\[
\frac{m_1 + g + s_1 \beta}{m_1 + g + s_1} > \frac{c_{h,2} + \beta(m_2 - m'_1 - s_1)}{m_2 - m'_1 - s_1 + c_{h,2}}, \quad (214)
\]

As the two linear functions of $\beta \frac{m_1 + g + s_1 \beta}{m_1 + g + s_1}$ and $\frac{c_{h,2} + \beta(m_2 - m'_1 - s_1)}{m_2 - m'_1 - s_1 + c_{h,2}}$ converge to the value 1 when $\beta$ converges to 1, existence of feasible $\beta$ values for condition (214) is equivalent to the following condition

\[
\frac{m_1 + g}{m_1 + g + s_1} > \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}}, \quad (215)
\]
Case B

In this situation, the following inequalities hold (see Figure (8)): \( \~a_{2,1}(t_1, t_2) \leq \~a_{2,3}(t_1, t_2) \leq \~a_{2,2}(t_1, t_2) \),

\[ \begin{align*}
\text{Region } C_{2,1} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}^2 \text{ and } \alpha \in ]\~a_{2,1}(t_1, t_2), \~a_{2,2}(t_1, t_2) [ \}, \\
\text{Region } C_{2,2} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}^2 \text{ and } \alpha \in ]\~a_{2,1}(t_1, t_2), \~a_{2,3}(t_1, t_2) [ \}, \\
\text{Region } C_{2,3} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}^2 \text{ and } \alpha \in ]\~a_{2,3}(t_1, t_2), \~a_{2,2}(t_1, t_2) [ \}, \\
\text{Region } C_{2,4} & = \{ (t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in \mathbb{R}^2 \text{ and } \alpha \in ]\~a_{2,2}(t_1, t_2), \infty [ \}.
\end{align*} \]

First order conditions by region.

\textbf{Region } C_{2,1}. \text{ In this region, the objective function is}
\[
I_{\beta,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1-\beta} \left[ (m_2 - m'_2) t_2 + s_1 (t_1 - t_2) \right] F(t_2) + (m'_2 - m_2) G(t_2) \\
+ (s_1 t_1 - c_0 t_2) (F(t_1) - F(t_2)) + (c_{0,2} - s_1) (G(t_1) - G(t_2)) \\
- ((m_1 + g) t_1 + c_0 t_2) (1 - F(t_1)) + (m_1 + c_{0,2} + g) (\mu - G(t_1)) - \alpha \right]
\]

(216)

and the first order derivatives are given by

\[
\frac{dI_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = -\frac{\beta}{(1-\beta)} < 0, \\
\frac{dI_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{(s_1 + m_1 + g) F(t_1) - (m_1 + g)}{(1-\beta)}, \\
\frac{dI_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m'_2 - s_1 + c_{0,2}) F(t_2) - c_{0,2}}{(1-\beta)}.
\]

(217)

(218)

(219)

**Region C_{2,2}**

According to Figure 8, let’s define \(T_1(\alpha, t_1, t_2)\) as the \(T\) value solving

\[
\alpha = -m'_1 (t_2 - T) + m_2 (t_2 - T) + s_1 (t_1 - t_2)
\]

(220)

and \(T_2(\alpha, t_1, t_2)\) as the \(T\) value solving

\[
\alpha = s_1 (t_1 - T) + c_0 (T - t_2)
\]

(221)

In this region, the objective function is

\[
I_{\beta,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1-\beta} \left[ (m_2 - m'_2) t_2 + s_1 (t_1 - t_2) - \alpha \right] F(T_1(\alpha, t_1, t_2)) \\
+ \frac{1}{1-\beta} [m'_2 - m_2] G(T_1(\alpha, t_1, t_2)) \\
+ \frac{1}{1-\beta} [s_1 t_1 - c_0 t_2 - \alpha] (F(t_1) - F(T_2(\alpha, t_1, t_2))) \\
+ (c_{0,2} - s_1) (G(t_1) - G(T_2(\alpha, t_1, t_2))) \\
+ \frac{1}{1-\beta} \left[ ((m_1 + g) t_1 + c_0 t_2 + \alpha) (1 - F(t_1)) + (m_1 + c_{0,2} + g) (\mu - G(t_1)) \right]
\]

(222)

and the first order derivatives are given by

\[
\frac{dI_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1 (F(T_1) + F(T_1(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2))) - (m_1 + g) (F(T_1) - 1)}{(1-\beta)}, \\
\frac{dI_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - s_1 - m'_2) F(T_1(\alpha, t_1, t_2)) - c_0 (1 - F(T_2(\alpha, t_1, t_2)))}{(1-\beta)}, \\
\frac{dI_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_2(\alpha, t_1, t_2)) - F(T_1(\alpha, t_1, t_2)) - \beta}{1-\beta}.
\]

(223)

(224)

(225)

**Region C_{2,3}**

According to Figure 8, let’s define \(T_3(\alpha, t_1, t_2)\) as the \(T\) value solving

\[
\alpha = m_1 (T - t_1) + g (T - t_1) + c_0 (T - t_2).
\]

(226)
In this region, the objective function is

\[ l_{\beta,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (s_1 t_1 - c_h t_2 - \alpha)(F(t_1) - F(T_2(\alpha, t_1, t_2))) ight. \\
\left. + (c_h - s_1)(G(t_1) - G(T_2(\alpha, t_1, t_2))) \right] + \frac{1}{1 - \beta} \left[ -((m_1 + g)t_1 + c_h t_2 + \alpha)(1 - F(t_1)) + (m_1 + c_h + g)(\mu - G(t_1)) \right] \]  

(227)

and the first order derivatives are given by:

\[ \frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1}{1 - \beta} (F(t_1) - F(T_2(t_1, t_2, \alpha))) - \frac{1}{1 - \beta} (m_1 + g)(1 - F(t_1)), \]  

(228)

\[ \frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{-c_h}{1 - \beta} (1 - F(T_2(t_1, t_2, \alpha))), \]  

(229)

\[ \frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_2(t_1, t_2, \alpha)) - \beta}{1 - \beta}. \]  

(230)

Region C_{2.4}.

In this region, the objective function is

\[ l_{\beta,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ -((m_1 + g)t_1 + c_h t_2 + \alpha)(1 - F(T_3(t_1, t_2, \alpha))) \right. \\
\left. + (m_1 + c_h + g)(\mu - G(T_3(t_1, t_2, \alpha))) \right]. \]  

(231)

and the first order derivatives are given by:

\[ \frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{-1}{1 - \beta} (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2))), \]  

(232)

\[ \frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{-c_h}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))), \]  

(233)

\[ \frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(\alpha, t_1, t_2)) - \beta}{1 - \beta}. \]  

(234)

Analysis of the first order conditions. It is direct to see that the only case where the first order conditions possibly have a solution is the region C_{2.2}. Under adequate assumptions, the first order conditions (223)-(225) have the solution

\[ t_{*,FOC}^{*,FOC} = F^{-1} \left( \frac{m_1 + g + s_1 \beta}{m_1 + g + s_1} \right), \]

(235)

\[ t_{*,FOC}^{*,FOC} = \left( \frac{m_2 - m_1'}{m_2 - m_1' - s_1 + c_h} \right) F^{-1} \left( \frac{c_h(1 - \beta)}{m_2 - m_1' - s_1 + c_h} \right) + \left( \frac{c_h - s_1}{m_2 - m_1' - s_1 + c_h} \right) F^{-1} \left( \frac{c_h + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_h} \right). \]

(236)

\[ s_{*,FOC}^{*,FOC} = \left( \frac{c_h - s_1}{m_2 - m_1' - s_1 + c_h} \right) F^{-1} \left( \frac{c_h + \beta(m_2 - m_1' - s_1)}{m_2 - m_1' - s_1 + c_h} \right) + s_{*,FOC}^{*,FOC} - c_h \]  

(237)

52
We also find the following parameter values

\[
T_1(t^{*,\text{FOC}}_{\beta,2,1}, t^{*,\text{FOC}}_{\beta,2,2}, \alpha_{\beta,2}^{*,\text{FOC}}) = F^{-1}\left(\frac{c_h(1-\beta)}{m_2-m_1'-s_1+c_h,2}\right),
\]

\[
T_2(t^{*,\text{FOC}}_{\beta,2,1}, t^{*,\text{FOC}}_{\beta,2,2}, \alpha_{\beta,2}^{*,\text{FOC}}) = F^{-1}\left(\frac{c_h,2+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h,2}\right).
\]

Several assumptions are required in order to guarantee that this solution belongs to the interior of \(C_{2,2}\):

\[
\frac{c_h,2+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h,2} < 1,
\]

\[
\frac{c_h(1-\beta)}{m_2-m_1'-s_1+c_h,2} < 1,
\]

\[
T^{*,\text{FOC}}_{\beta,2,2} < t^{*,\text{FOC}}_{\beta,2,1},
\]

\[
\hat{a}_{2,1}(t^{*,\text{FOC}}_{\beta,2,1}, t^{*,\text{FOC}}_{\beta,2,2}) < a_{\beta,2}^{*,\text{FOC}} < \hat{a}_{2,2}(t^{*,\text{FOC}}_{\beta,2,1}, t^{*,\text{FOC}}_{\beta,2,2}).
\]

From Figure (8) it can be seen that the last condition is equivalent to

\[
0 \leq T_1(t^{*,\text{FOC}}_{\beta,2,1}, t^{*,\text{FOC}}_{\beta,2,2}, \alpha_{\beta,2}^{*,\text{FOC}}) \leq T_2(t^{*,\text{FOC}}_{\beta,2,1}, t^{*,\text{FOC}}_{\beta,2,2}, \alpha_{\beta,2}^{*,\text{FOC}}) \leq t^{*,\text{FOC}}_{\beta,2,1}.
\]

Condition (240) is equivalent to \(m_2-m_1'-s_1 \geq 0\). Conditions (244)-(245) thus amount to

\[
\frac{m_1+g+s_1\beta}{m_1+g+s_1} > \frac{c_h,2+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h,2}
\]

As the two linear functions of \(\beta \frac{m_1+g+s_1\beta}{m_1+g+s_1}\) and \(\frac{c_h,2+\beta(m_2-m_1'-s_1)}{m_2-m_1'-s_1+c_h,2}\) converge to the value 1 when \(\beta\) converges to 1, existence of feasible \(\beta\) values for condition (247) is equivalent to the following condition

\[
\frac{m_1+g}{m_1+g+s_1} > \frac{c_h,2}{m_2-m_1'-s_1+c_h,2}.
\]
**Property 4.b:** Consider the setting \( c_{h,2} \leq s_1 \). Under the assumption

\[
\begin{align*}
    m_2 - m'_1 - s_1 + c_{h,2} &> 0, \quad (248) \\
    m_2 - m'_1 - s_1 &< c_{h,2}, \quad (249) \\
    \frac{m_1 + g}{m_1 + g + s_1} &> \frac{c_{h,2}}{m_2 - m'_1 - s_1 + c_{h,2}} \quad (250)
\end{align*}
\]

and for \( \beta \) values satisfying

\[
\beta > \frac{m_2 - m'_1 - s_1}{c_{h,2}}, \quad (251)
\]

the first-order conditions solutions

\[
\begin{align*}
    t^*_\beta,F_{OC,1}^{,1} &= \left( \frac{m_1 + g + c_{h,2}}{m_1 + g + s_1} \right)^{-1} \frac{m_1 + g + \beta s_1}{m_1 + g + s_1} \\
    t^*_\beta,F_{OC,2} &= \left( \frac{s_1 - c_{h,2}}{m_1 + g + s_1} \right)^{-1} \frac{(m_1 + g)(1 - \beta)}{m_1 + g + s_1} \\
    t^*_\beta,F_{OC,3} &= \frac{c_{h,2}(1 - \beta)}{m_2 - m'_1 - s_1 + c_{h,2}} \quad (253)
\end{align*}
\]

correspond to the unique finite minimum of the CVaR loss function \( l_{\beta,2}(\cdot,\cdot,\cdot) \) in \( \mathbb{R}_2 \). While if if condition (248), (249), (250) or (251) is not satisfied, there is no finite minimum in \( \mathbb{R}_2 \) for \( l_{\beta,2}(\cdot,\cdot,\cdot) \).

**Proof.** According to equation (4), it can be seen that the critical values for the \( \alpha \) parameters corresponding to the slope discontinuities for the piecewise linear function \( L_2(t_1,t_2,\cdot) \), as functions of \( t_1 \) and \( t_2 \), are given by

\[
\begin{align*}
    \tilde{\alpha}_{2,1}(t_1,t_2) &= s_1(t_1 - t_2), \quad (255) \\
    \tilde{\alpha}_{2,2}(t_1,t_2) &= (m_2 - m'_1)t_2 + s_1(t_1 - t_2), \quad (256) \\
    \tilde{\alpha}_{2,3}(t_1,t_2) &= c_{h}(t_1 - t_2). \quad (257)
\end{align*}
\]

The ordering of these critical values conditions the structure of the optimality conditions. In the considered setting, as \( c_{h,2} \leq s_1 \), one has the unique following ordering \( \tilde{\alpha}_{2,3}(t_1,t_2) \leq \tilde{\alpha}_{2,1}(t_1,t_2) \leq \tilde{\alpha}_{2,2}(t_1,t_2) \) (see Figure (9)).
In order to characterize the first order conditions, we define the regions $C_{2,1}$, $C_{2,2}$, $C_{2,3}$ and $C_{2,4}$, as

\[ C_{2,1} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_2 \text{ and } \alpha \in ]\infty, \tilde{\alpha}_{2,3}(t_1, t_2)[, \]

\[ C_{2,2} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_2 \text{ and } \alpha \in ]\tilde{\alpha}_{2,3}(t_1, t_2), \tilde{\alpha}_{2,1}(t_1, t_2)[, \]

\[ C_{2,3} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_2 \text{ and } \alpha \in ]\tilde{\alpha}_{2,1}(t_1, t_2), \tilde{\alpha}_{2,2}(t_1, t_2)[, \]

\[ C_{2,4} = \{(t_1, t_2, \alpha) \text{ with } (t_1, t_2) \in R_2 \text{ and } \alpha \in ]\tilde{\alpha}_{2,2}(t_1, t_2), \infty[, \]

**First order conditions by region.**

**Region $C_{2,1}$.** In this region, the objective function is

\[
l_{2,3}(t_1, t_2, \alpha) = \alpha + \frac{1}{1-\beta} \left[ \left( (m_2 - m_1')t_2 + s_1(t_1 - t_2) \right) F(t_2) + (m_1' - m_2)G(t_2) \\
+ (s_1t_1 + c_1t_2)(F(t_1) - F(t_2)) + (c_{h,2} - s_1)(G(t_1) - G(t_2)) \\
- ((m_1 + g)t_1 + c_1t_2)(1 - F(t_1)) + (m_1 + c_{h,2} + g)(\mu - G(t_1)) - \alpha \right].
\]

and the first order derivatives are given by
According to Figure 9, let's define $T_2(\alpha, t_1, t_2)$ as the $T$ value solving

$$\alpha = s_1(t_1 - T) + c_h(T - t_2)$$

(262)

and $T_3(\alpha, t_1, t_2)$ as the $T$ value solving

$$\alpha = m_1(T - t_1) + g(T - t_1) + c_h(T - t_2).$$

(263)

In this region, the objective function is

$$l_{\beta,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_2 - m_1' - s_1 + c_h,2)t_2 F(t_2) + (c_h,2 - s_1)G(T_2(t_1, t_2, \alpha)) 
- (m_2 - m_1' - s_1 + c_h,2)G(t_2) + (s_1 t_1 - c_h,2 - \alpha)F(T_2(t_1, t_2, \alpha)) 
- ((m_1 + g)t_1 + c_h,2 + \alpha)(1 - F(T_3(t_1, t_2, \alpha))) 
+ (m_1 + c_h,2 + g)(\mu - G(T_3(t_1, t_2, \alpha)))) \right].$$

(264)

and the first order derivatives are given by:

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1F(T_2(\alpha, t_1, t_2)) - (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2)))}{1 - \beta},$$

(265)

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1 + c_h,2)F(t_2) - c_h,2F(T_2(\alpha, t_1, t_2)) - c_h(1 - F(T_3(\alpha, t_1, t_2)))}{1 - \beta},$$

(266)

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2)) - \beta}{1 - \beta}. $$

(267)

Region $C_{2,3}$.

According to Figure 9, let's define $T_1(\alpha, t_1, t_2)$ as the $T$ value solving

$$\alpha = -m_1'(t_2 - T) + m_2(t_2 - T) + s_1(t_1 - t_2)$$

(268)

In this region, the objective function is

$$l_{\beta,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1 - \beta} \left[ (m_1' - m_2)G(T_1(t_1, t_2, \alpha)) + ((m_2 - m_1' - s_1)t_2 + s_1 t_1 - \alpha)F(T_1(t_1, t_2, \alpha)) 
+ ((m_1 + g)t_1 + c_h,2 + \alpha)(1 - F(T_3(t_1, t_2, \alpha))) 
+ (m_1 + c_h,2 + g)(\mu - G(T_3(t_1, t_2, \alpha)))) \right].$$

(269)

and the first order derivatives are given by:

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_1} = \frac{s_1F(T_1(\alpha, t_1, t_2)) - (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2)))}{1 - \beta},$$

(270)

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 - m_1' - s_1)F(T_1(\alpha, t_1, t_2)) - c_h(1 - F(T_3(\alpha, t_1, t_2)))}{1 - \beta},$$

(271)

$$\frac{dl_{\beta,2}(t_1, t_2, \alpha)}{d\alpha} = \frac{F(T_3(\alpha, t_1, t_2)) - F(T_1(\alpha, t_1, t_2)) - \beta}{1 - \beta}. $$

(272)
Region $C_{2,4}$. 
In this region, the objective function is

$$I_{3,2}(t_1, t_2, \alpha) = \alpha + \frac{1}{1-\beta} \left\{ -((m_1+g)t_1 + c_h t_2 + \alpha) (1 - F(T_3(t_1, t_2, \alpha)) \right. $$

$$\left. + \,(m_1 + c_h,2 + g)(\mu - G(T_3(t_1, t_2, \alpha))) \right\].$$

(273) 

and the first order derivatives are given by:

$$\frac{dt_{3,2}(t_1, t_2, \alpha)}{dt_1} = -1 \frac{1}{1-\beta} (m_1 + g)(1 - F(T_3(\alpha, t_1, t_2))),$$

(274) 

$$\frac{dt_{3,2}(t_1, t_2, \alpha)}{dt_2} = -\frac{ch_2}{1-\beta} (1 - F(T_3(\alpha, t_1, t_2))),$$

(275) 

$$\frac{dt_{3,2}(t_1, t_2, \alpha)}{d\alpha} = F(T_3(\alpha, t_1, t_2)) - \beta.$$

(276) 

Analysis of the first order conditions. It is direct to see that the only case where the first order conditions possibly have a solution is the region $C_{2,2}$. Under adequate assumptions, the first order conditions (265)-(267) have the solution

$$t_{3,2,1}^{*, FOC} = \left(\frac{m_1+g+c_h,2}{m_1+g+s_1}\right) F^{-1}\left(\frac{m_1+g+s_1(1-\beta)}{m_1+g+s_1}(m_1+g)(1-\beta)\right),$$

(277) 

$$t_{3,2,2}^{*, FOC} = F^{-1}\left(\frac{c_h(1-\beta)}{m_2-m_1-1+s_1+c_h,2}\right),$$

(278) 

$$a_{3,2}^{*, FOC} = (c_h,2-s_1) F^{-1}\left(\frac{m_1+g+s_1(1-\beta)}{m_1+g+s_1}\right) + s_1 t_{3,2,1}^{*, FOC} + c_h t_{3,2,2}^{*, FOC}.$$

(279) 

We also find the following parameter values

$$T_2(t_{3,2,1}^{*, FOC}, t_{3,2,2}^{*, FOC}, a_{3,2}^{*, FOC}) = F^{-1}\left(\frac{m_1+g+s_1\beta}{m_1+g+s_1}\right),$$

(280) 

$$T_3(t_{3,2,1}^{*, FOC}, t_{3,2,2}^{*, FOC}, a_{3,2}^{*, FOC}) = F^{-1}\left(\frac{m_1+g+s_1\beta}{m_1+g+s_1}\right),$$

(281) 

Now, several assumptions are required in order to guarantee that this solution belongs to the interior of $C_{2,2}$. Basically these assumptions are the following

$$\frac{c_h(1-\beta)}{m_2-m_1-1+s_1+c_h,2} < 1,$$

(282) 

$$t_{3,2,2}^{*, FOC} < t_{3,2,1}^{*, FOC},$$

(283) 

$$a_{3,2,1}^{*, FOC} < a_{3,2}^{*, FOC} < \alpha_2,3(t_{3,2,1}^{*, FOC}, t_{3,2,2}^{*, FOC}).$$

(284) 

From Figure 9 it can be seen that the last condition is equivalent to

$$t_{3,2,2}^{*, FOC} \leq T_2(t_{3,2,1}^{*, FOC}, t_{3,2,2}^{*, FOC}, a_{3,2}^{*, FOC}) \leq t_{3,2,1}^{*, FOC},$$

(285) 

$$t_{3,2,2}^{*, FOC} \leq T_3(t_{3,2,1}^{*, FOC}, t_{3,2,2}^{*, FOC}, a_{3,2}^{*, FOC}).$$

(286)
Condition (282) is equivalent to
\[ \beta > \frac{m_2 - m_1' - s_1}{c_{b,2}}. \] (287)

Conditions (285)-(286) thus amount to
\[ \frac{m_1 + g}{m_1 + g + s_1} > \frac{c_{b,2}}{m_2 - m_1' - s_1 + c_{b,2}}. \] (288)

D APPENDIX B-3: First Order Conditions for \( l_{\beta,b}(t, \alpha) \)

Property 5: Under assumption (9), the boundary loss function \( l_{\beta,b}(\cdot, \cdot) \) has a unique finite minimum over \( \mathbb{R}^+ \) given by
\[
\tilde{t}^{*, FOC}_{\beta,b} = \left( \frac{m_2 - m_1'}{m_2 - m_1' + m_1 + c_{b,2} + g} \right) F^{-1} \left( \frac{(m_1 + c_{b,2} + g)(1 - \beta)}{m_2 - m_1' + m_1 + c_{b,2} + g} \right) + \left( \frac{m_1 + c_{b,2} + g}{m_2 - m_1' + m_1 + c_{b,2} + g} \right) F^{-1} \left( \frac{m_1 + c_{b,2} + g + \beta(m_2 - m_1')}{m_2 - m_1' + m_1 + c_{b,2} + g} \right). \] (289)

Proof. In this case the objective function is thus given by
\[
l_{\beta,b}(t, \alpha) = \alpha + \frac{1}{1-\beta} \left( \int_0^t [m_1(T-t) + m_2(t-T) - \alpha]^+ f(T)dT \right)
+ \int_t^\infty [m_1(T-t) + (c_{b,2} + g)(T-t) - \alpha]^+ f(T)dT. \] (290)

According to equation (3), it can be seen that the critical values for the \( \alpha \) parameters corresponding to the slope discontinuities for the piecewise linear function (6), as functions of \( t \) are given by
\[
\tilde{\alpha}_{b,1}(t) = 0, \] (291)
\[
\tilde{\alpha}_{b,2}(t) = (m_2 - m_1')t, \] (292)
with \( \tilde{\alpha}_{b,1}(t) \leq \tilde{\alpha}_{b,2}(t) \) (see Figure (10)),

58
We define the regions \( C_{b,1} \), \( C_{b,2} \) and \( C_{b,3} \), defined as

\[
\begin{align*}
C_{b,1} &= \{(t, \alpha) \mid t \in \mathbb{R}^+ \text{ and } \alpha \in ]\infty, \tilde{\alpha}_{b,1}(t, t)[, \\
C_{b,2} &= \{(t, \alpha) \mid t \in \mathbb{R}^+ \text{ and } \alpha \in ]\tilde{\alpha}_{b,1}(t), \tilde{\alpha}_{b,2}(t)[, \\
C_{b,3} &= \{(t, \alpha) \mid t \in \mathbb{R}^+ \text{ and } \alpha \in ]\tilde{\alpha}_{b,2}(t), \infty[, 
\end{align*}
\]

Expression of the first order conditions.

Region \( C_{b,1} \).

In this region, the objective function is

\[
L_{\beta}(t, \alpha) = \alpha + \frac{1}{1-\beta} \left[ (m_2 t - m'_1 t) F(t) + (m'_1 - m_2) G(t) \\
+ m_1 (\mu - G(t) - t(1 - F(t))) \\
+ (c_{2,1} + \alpha)(\mu - G(t) - t(1 - F(t))) - \alpha \right].
\]

and the first order derivatives are given by

\[
\begin{align*}
\frac{dL_{\beta}(t, \alpha)}{dt} &= \frac{(m_1 - m'_1 + m_2 + c_{2,1} + \alpha)}{1-\beta} F(t) - \frac{(m_1 + g + c_{2,2})}{1-\beta}, \\
\frac{dL_{\beta}(t, \alpha)}{d\alpha} &= -\beta 1-\beta < 0.
\end{align*}
\]

Region \( C_{b,2} \).

According to Figure 4, let’s define \( T_1(\alpha, t) \) as the \( T \) value solving

\[
\alpha = (m_2 - m'_1)(t - T),
\]
and $T_2(\alpha, t)$ as the $T$ value solving
\[ \alpha = (m_1 + g + c_{h,2})(T - t). \]  
(297)

The objective function is
\[ l_{\beta, b}(t, \alpha) = \alpha + \frac{1}{1-\beta} \left[ (m_1 - m_2)G(T_1(t, \alpha)) + (m_1 + c_{h,2} + g)(\mu - G(T_2(t, \alpha))) \right. \]
\[ + \left. (-m'_1 t + m_2 t - \alpha)F(T_1(t, \alpha)) \right. \]
\[ + \left. (-m_1 t - g t - c_{h,2} t - \alpha)(1 - F(T_2(t, \alpha))) \right], \]  
(298)

and the first order derivatives are given by:
\[ \frac{dl_{\beta, b}(t, \alpha)}{dt} = \frac{(m_2 - m'_1)}{1-\beta} F(T_1(t, \alpha)) - \frac{(m_1 + g + c_{h,2})}{1-\beta} (1 - F(T_2(t, \alpha))). \]  
(299)
\[ \frac{dl_{\beta, b}(t, \alpha)}{da} = F(T_2(t, \alpha)) - F(T_1(t, \alpha)) - \beta. \]  
(300)

**Region C_{h,3}.**

The objective function is
\[ l_{\beta, b}(t, \alpha) = \alpha + \frac{1}{1-\beta} \left[ (m_1 + c_{h,2} + g)(\mu - G(T_2(t, \alpha))) \right. \]
\[ - \left. ((m_1 + g + c_{h,2})t + \alpha)(1 - F(T_2(t, \alpha))) \right]. \]  
(301)

and the first order derivatives are given by:
\[ \frac{dl_{\beta, b}(t, \alpha)}{dt} = \frac{(m_1 + g + c_{h,2})}{1-\beta} \left(1 - F(T_2(t, \alpha))\right) < 0, \]  
(302)
\[ \frac{dl_{\beta, b}(t, \alpha)}{da} = \frac{F(T_2(t, \alpha)) - \beta}{1-\beta}. \]  
(303)

**Analysis of the first order conditions.** It is direct to see that the only case where the first order conditions possibly have a solution is the region $C_{h,2}$. Under adequate assumptions, the first order conditions (299)-(300) have the solution

\[ t_{\beta, b}^{*, FOC} = \left( \frac{m_2 - m'_1}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right) F^{-1} \left( \frac{(m_1 + c_{h,2} + g)(1 - \beta)}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right) \]
\[ + \left( \frac{m_1 + c_{h,2} + g}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right) F^{-1} \left( \frac{(m_1 + c_{h,2} + g + \beta m_{2} - m'_1)}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right), \]  
(304)
\[ a_{\beta, b}^{*, FOC} = \frac{(m_2 - m'_1)}{m_2 - m'_1 + m_1 + c_{h,2} + g} - \frac{(m_2 - m'_1)}{m_2 - m'_1 + m_1 + c_{h,2} + g} F^{-1} \left( \frac{(m_1 + c_{h,2} + g)(1 - \beta)}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right). \]  
(305)

We also find the following parameter values
\[ T_1(t_{\beta, b}^{*, FOC}, a_{\beta, b}^{*, FOC}) = F^{-1} \left( \frac{(m_1 + c_{h,2} + g)(1 - \beta)}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right), \]  
(306)
\[ T_2(t_{\beta, b}^{*, FOC}, a_{\beta, b}^{*, FOC}) = F^{-1} \left( \frac{(m_1 + c_{h,2} + g + \beta m_{2} - m'_1)}{m_2 - m'_1 + m_1 + c_{h,2} + g} \right). \]  
(307)
Now, assumptions are required in order to guarantee that this solution belongs to the interior of $C_{b,2}$. Basically these assumptions are the following

$$\hat{a}_{b,1}(t^*_{\beta,b}^{FOC}) < a^*_{\beta,b}^{FOC} \leq \hat{a}_{b,2}(t^*_{\beta,b}^{FOC}).$$

(308)

From Figure 9 it can be seen that the last condition is equivalent to

$$0 \leq T_1(t^*_{\beta,b}^{FOC}, a^*_{\beta,b}^{FOC}) \leq t^*_{\beta,b}^{FOC},$$

(309)

$$t^*_{\beta,b}^{FOC} \leq T_2(t^*_{\beta,b}^{FOC}, a^*_{\beta,b}^{FOC}),$$

(310)

which is guaranteed to be satisfied.

### E APPENDIX C: Differentiability of the CVaR Function

Differentiability of the CVaR loss function $l_{\beta,1}(t_1, t_2, \alpha)$ inside $R_1 \times R$

**Case 1 : $m_2 \geq m_1 \geq m'_1$**

According to the 4 regions, the expressions of the the first order conditions of $l_{\beta,1}(t_1, t_2, \alpha)$ are as follows:

**Region C$_{1,1}$.**

The first order derivatives are given by

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{da} = -\frac{\beta}{1-\beta},$$

(311)

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_1 - m'_1)}{1-\beta} F(t_1) - \frac{(m_1 + g)}{1-\beta},$$

(312)

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + c_{a,2} + g)}{1-\beta} F(t_2) - \frac{c_{a,2}}{1-\beta},$$

(313)

**Region C$_{1,2}$.**

The first order derivatives are given by:

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_1 - m'_1)}{1-\beta} F(t_1) - \frac{(m_1 + g)}{1-\beta} F(T_2(\alpha, t_1, t_2)) - \frac{(m_1 + g)}{1-\beta} (1 - F(T_3(\alpha, t_1, t_2)),$$

(314)

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1-\beta} F(T_2(\alpha, t_1, t_2)) - \frac{c_{a,2}}{1-\beta} (1 - F(T_3(\alpha, t_1, t_2)),$$

(315)

$$\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{da} = \frac{F(T_3(t_1, t_2, \alpha)) - F(T_2(t_1, t_2, \alpha)) - \beta}{1-\beta}.$$
Region C\(_{1.3}\).

The first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = -\frac{(m_1' + g)}{1 - \beta} (1 - F(T_1(\alpha, t_1, t_2))) - \frac{(m_1 + g)}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))),
\]
\[
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = -\frac{(m_2 + g)}{1 - \beta} (1 - F(T_1(\alpha, t_1, t_2))) - \frac{c_{\alpha,2}}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))),
\]
\[
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = -\frac{F(T_3(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha)) - \beta}{1 - \beta}.
\]

Region C\(_{1.4}\).

The first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_1} = -(m_1 + g)(1 - F(T_3(t_1, t_2, \alpha))),
\]
\[
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{dt_2} = -\frac{c_{\alpha,2}}{1 - \beta} (1 - F(T_3(t_1, t_2, \alpha))),
\]
\[
\frac{dl_{\beta,1}(t_1,t_2,\alpha)}{d\alpha} = \frac{F(T_3(t_1, t_2, \alpha) - \beta}{1 - \beta}.
\]

From the expression of the first order derivatives in the 4 regions, it is clear that \(l_{\beta,1}(t_1,t_2,\alpha)\) is differentiable within each region, yet we have to examine the differentiability at the critical points of \(\tilde{\alpha}_{1,1}(t_1,t_2), \tilde{\alpha}_{1,2}(t_1,t_2), \) and \(\tilde{\alpha}_{1,3}(t_1,t_2)\) as follows:

**Differentiability at \(\tilde{\alpha}_{1,1}(t_1,t_2)\)**

If \(\alpha \to \tilde{\alpha}_{1,1}^+(t_1,t_2)\), we are in region C\(_{1.1}\) and the first order derivatives are given by the expressions (311-313).

If \(\alpha \to \tilde{\alpha}_{1,1}^-(t_1,t_2)\), we are in region C\(_{1.2}\) and the first order derivatives are given by the expressions (314-316).

Yet for \(\alpha = \tilde{\alpha}_{1,1}^+(t_1,t_2)\), one gets \(T_2(t_1,t_2,\alpha) = T_3(t_1,t_2,\alpha) = t_2\), and thus the first order derivatives of region C\(_{1.1}\) become equal to that of region C\(_{1.2}\) and the function is differentiable at \(\tilde{\alpha}_{1,1}(t_1,t_2)\).

**Differentiability at \(\tilde{\alpha}_{1,2}(t_1,t_2)\)**

If \(\alpha \to \tilde{\alpha}_{1,2}^+(t_1,t_2)\), we are in region C\(_{1.2}\) and the first order derivatives are given by the expressions (314-316).

If \(\alpha \to \tilde{\alpha}_{1,2}^-(t_1,t_2)\), we are in region C\(_{1.3}\) and and the first order derivatives are given
by the expressions (317-319).
Yet for $\alpha = \tilde{\alpha}_{1,2}(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = T_2(t_1, t_2, \alpha) = t_1$, and thus the first order derivatives of region $C_{1,2}$ become equal to that of region $C_{1,3}$ and the function is differentiable at $\tilde{\alpha}_{1,2}(t_1, t_2)$.

Differentiability at $\tilde{\alpha}_{1,3}(t_1, t_2)$
If $\alpha \rightarrow \tilde{\alpha}_{1,3}^-(t_1, t_2)$, we are in region $C_{1,3}$ and the first order derivatives are given by the expressions (317-319).
If $\alpha \rightarrow \tilde{\alpha}_{1,3}^+(t_1, t_2)$, we are in region $C_{1,4}$ and the first order derivatives are given by the expressions (320-322).
Yet for $\alpha = \tilde{\alpha}_{1,3}^+(t_1, t_2)$, one gets $T_1(t_1, t_2, \alpha) = 0$, and thus the first order derivatives of region $C_{1,3}$ become equal to that of region $C_{1,4}$ and the function is differentiable at $\tilde{\alpha}_{1,3}(t_1, t_2)$.

Case 2: $m_1 \geq m_2 \geq m'_1$

Case 2.a

- $\tilde{\alpha}_{1,1}(t_1, t_2) \leq \tilde{\alpha}_{1,2}(t_1, t_2) \leq \tilde{\alpha}_{1,3}(t_1, t_2)$

From the expression of the first order derivatives in the 4 regions, it is clear that $l_{1,1}(t_1, t_2, \alpha)$ is differentiable within each region, yet we have to examine the differentiability at the critical points of $\tilde{\alpha}_{1,1}(t_1, t_2)$, $\tilde{\alpha}_{1,2}(t_1, t_2)$, and $\tilde{\alpha}_{1,3}(t_1, t_2)$ as follows:

According to the 4 regions, the expressions of the first order conditions of $l_{1,1}(t_1, t_2, \alpha)$ are as follows:

Region $C_{1,1}$. The first order derivatives are given by

$$\frac{dl_{1,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{-\beta}{1-\beta},$$

$$\frac{dl_{1,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_1 - m'_1)}{1-\beta} F(t_1) - \frac{(m_1 + g)}{1-\beta},$$

$$\frac{dl_{1,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + ch_{h,2} + g)}{1-\beta} F(t_2) - \frac{ch_{h,2}}{1-\beta}.$$ 

Region $C_{1,2}$.
The first order derivatives are given by:
\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{(m_1' + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) + \frac{(m_1 + g)}{1 - \beta} F(T_2(\alpha, t_1, t_2))
\]

(326)

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_1 + g)}{1 - \beta},
\]

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{(m_2 + g)}{1 - \beta} (F(T_1(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2))) - \frac{c_{h, 2}}{1 - \beta} (1 - F(t_2)) + \frac{(m_1 + g)}{1 - \beta} F(t_2).
\]

(327)

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = F(T_2(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha)) - \beta.
\]

(328)

**Region C_{1.3}**.

The first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{(m_1' + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) - \frac{(m_1 + g)}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2))).
\]

(329)

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = \frac{(m_2 + g)}{1 - \beta} (F(T_1(\alpha, t_1, t_2)) - F(T_3(\alpha, t_1, t_2))) - \frac{c_{h, 2}}{1 - \beta} (1 - F(T_3(\alpha, t_1, t_2)) - \frac{m_1 + g}{1 - \beta} F(T_3(\alpha, t_1, t_2)).
\]

(330)

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{F(T_3(t_1, t_2, \alpha)) - F(T_3(t_1, t_2, \alpha)) - \beta}{1 - \beta}.
\]

(331)

**Region C_{1.4}**.

The first order derivatives are given by:

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{d\alpha} = \frac{(m_1' + g)}{1 - \beta} (1 - F(T_3(t_1, t_2, \alpha))).
\]

(332)

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_1} = -\frac{(m_1 + g)}{1 - \beta} (1 - F(T_3(t_1, t_2, \alpha))).
\]

(333)

\[
\frac{dl_{\beta,1}(t_1, t_2, \alpha)}{dt_2} = \frac{F(T_3(t_1, t_2, \alpha)) - F(T_3(t_1, t_2, \alpha)) - \beta}{1 - \beta}.
\]

(334)

**Differentiability at \( \tilde{\alpha}_{1,1}(t_1, t_2) \)**

If \( \alpha \rightarrow \tilde{\alpha}_{1,1}^{-1}(t_1, t_2) \), we are in region C_{1.1} and the first order derivatives are given by the expressions (323-325).

If \( \alpha \rightarrow \tilde{\alpha}_{1,1}^{+}(t_1, t_2) \), we are in region C_{1.2} and the first order derivatives are given by the expressions (326-328).

Yet for \( \alpha = \tilde{\alpha}_{1,1}^{+}(t_1, t_2) \), one gets \( T_1(t_1, t_2, \alpha) = T_2(t_1, t_2, \alpha) = t_1 \), and thus the first order derivatives of region C_{1.1} become equal to that of region C_{1.2} and the function is differentiable at \( \tilde{\alpha}_{1,1}(t_1, t_2) \).

**Differentiability at \( \tilde{\alpha}_{1,2}(t_1, t_2) \)**
If \( \alpha \rightarrow \tilde{\alpha}_{1,2}(t_1, t_2) \), we are in region \( C_{1,2} \) and the first order derivatives are given by the expressions (326-328).

If \( \alpha \rightarrow \tilde{\alpha}_{1,2}^+(t_1, t_2) \), we are in region \( C_{1,3} \) and the first order derivatives are given by the expressions (329-331).

Yet for \( \alpha = \tilde{\alpha}_{1,2}^+(t_1, t_2) \), one gets \( T_2(t_1, t_2, \alpha) = T_3(t_1, t_2, \alpha) = t_2 \), and thus the first order derivatives of region \( C_{1,2} \) become equal to that of region \( C_{1,3} \) and the function is differentiable at \( \tilde{\alpha}_{1,2}(t_1, t_2) \).

**Differentiability at \( \tilde{\alpha}_{1,3}(t_1, t_2) \)**

If \( \alpha \rightarrow \tilde{\alpha}_{1,3}^+(t_1, t_2) \), we are in region \( C_{1,3} \) and the first order derivatives are given by the expressions (329-331).

If \( \alpha \rightarrow \tilde{\alpha}_{1,3}^+(t_1, t_2) \), we are in region \( C_{1,4} \) and the first order derivatives are given by the expressions (123-346).

Yet for \( \alpha = \tilde{\alpha}_{1,3}^+(t_1, t_2) \), one gets \( T_1(t_1, t_2, \alpha) = 0 \), and thus the first order derivatives of region \( C_{1,3} \) become equal to that of region \( C_{1,4} \) and the function is differentiable at \( \tilde{\alpha}_{1,3}(t_1, t_2) \).

**Case 2.b** In order to characterize the first order conditions, we define the regions for

- \( \tilde{\alpha}_{1,1}(t_1, t_2) \leq \tilde{\alpha}_{1,3}(t_1, t_2) \leq \tilde{\alpha}_{1,2}(t_1, t_2) \).

From the expression of the first order derivatives in the 4 regions, it is clear that \( l_{1,1}(t_1, t_2, \alpha) \) is differentiable within each region, yet we have to examine the differentiability at the critical points of \( \tilde{\alpha}_{1,1}(t_1, t_2) \), \( \tilde{\alpha}_{1,2}(t_1, t_2) \), and \( \tilde{\alpha}_{1,3}(t_1, t_2) \) as follows:

According to the 4 regions, the expressions of the the first order conditions of \( l_{1,1}(t_1, t_2, \alpha) \) are as follows:

**Region \( C_{1,1} \).**

The first order derivatives are given by
\[
\begin{align*}
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_1} &= \frac{-\beta}{1 - \beta}, \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_1 - m_2')}{1 - \beta} F(t_1) - \frac{(m_1 + g)}{1 - \beta}, \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dx} &= \frac{(m_2 + c_{h,2} + g)}{1 - \beta} F(t_2) - \frac{c_{h,2}}{1 - \beta},
\end{align*}
\]

### Region C_{1,2}.

The first order derivatives are given by:

\[
\begin{align*}
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_1} &= -\frac{(m_1 + g)}{1 - \beta} F(T_1(\alpha, t_1, t_2)) + \frac{(m_1 + g)}{1 - \beta} F(T_2(\alpha, t_1, t_2)) \\
& \quad - \frac{(m_1 + g)}{1 - \beta}, \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + g)}{1 - \beta} (F(T_1(\alpha, t_1, t_2)) - F(T_2(\alpha, t_1, t_2))) - \frac{c_{h,2}}{1 - \beta} (1 - F(t_2)) + \frac{(m_1 + g)}{1 - \beta} F(t_2), \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dx} &= F(T_2(t_1, t_2, \alpha)) - F(T_1(t_1, t_2, \alpha)) - \frac{\beta}{1 - \beta}.
\end{align*}
\]

### Region C_{1,3}.

The first order derivatives are given by:

\[
\begin{align*}
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_1} &= -\frac{(m_1 + g)}{1 - \beta} (F(t_2) - F(T_2(\alpha, t_1, t_2))) - \frac{(m_1 + g)}{1 - \beta} (1 - F(t_2)), \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_2} &= \frac{(m_2 + g)}{1 - \beta} (F(t_2) - F(T_2(\alpha, t_1, t_2))) - \frac{c_{h,2}}{1 - \beta} (1 - F(t_2)), \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dx} &= F(T_2(t_1, t_2, \alpha)) - \frac{\beta}{1 - \beta}.
\end{align*}
\]

### Region C_{1,4}.

The first order derivatives are given by:

\[
\begin{align*}
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_1} &= \left(\frac{m_1 + g}{1 - \beta}\right) \left(1 - F(T_3(t_1, t_2, \alpha))\right), \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dt_2} &= \left(\frac{c_{h,2}}{1 - \beta}\right) \left(1 - F(T_3(t_1, t_2, \alpha))\right), \\
\frac{dl_{j,1}(t_1, t_2, \alpha)}{dx} &= F(T_3(t_1, t_2, \alpha)) - \frac{\beta}{1 - \beta}.
\end{align*}
\]

### Differentiability at $\tilde{\alpha}_{1,1}(t_1, t_2)$

If $\alpha \to \tilde{\alpha}_{1,1}(t_1, t_2)$, we are in region C_{1,1} and the first order derivatives are given by
the expressions (335-337).

If \( \alpha \to \alpha_{1,1}^+(t_1, t_2) \), we are in region \( \mathbf{C}_{1,2} \) and and the first order derivatives are given by the expressions (338-340).

Yet for \( \alpha = \alpha_{1,1}^+(t_1, t_2) \), one gets \( T_1(t_1, t_2, \alpha) = T_2(t_1, t_2, \alpha) = t_1 \), and thus the first order derivatives of region \( \mathbf{C}_{1,1} \) become equal to that of region \( \mathbf{C}_{1,2} \) and the function is differentiable at \( \alpha_{1,1}(t_1, t_2) \).

**Differentiability at \( \alpha_{1,3}(t_1, t_2) \)**

If \( \alpha \to \alpha_{1,3}^+(t_1, t_2) \), we are in region \( \mathbf{C}_{1,2} \) and and the first order derivatives are given by the expressions (338-340).

If \( \alpha \to \alpha_{1,3}^-(t_1, t_2) \), we are in region \( \mathbf{C}_{1,3} \) and and the first order derivatives are given by the expressions (341-343).

Yet for \( \alpha = \alpha_{1,2}^+(t_1, t_2) \), one gets \( T_1(t_1, t_2, \alpha) = 0 \), and thus the first order derivatives of region \( \mathbf{C}_{1,2} \) become equal to that of region \( \mathbf{C}_{1,3} \) and the function is differentiable at \( \alpha_{1,3}(t_1, t_2) \).

**Differentiability at \( \alpha_{1,2}(t_1, t_2) \)**

If \( \alpha \to \alpha_{1,2}^+(t_1, t_2) \), we are in region \( \mathbf{C}_{1,3} \) and and the first order derivatives are given by the expressions (341-343).

If \( \alpha \to \alpha_{1,2}^-(t_1, t_2) \), we are in region \( \mathbf{C}_{1,4} \) and and the first order derivatives are given by the expressions (344-346).

Yet for \( \alpha = \alpha_{1,3}^+(t_1, t_2) \), one gets \( T_2(t_1, t_2, \alpha) = T_3(t_1, t_2, \alpha) = t_2 \), and thus the first order derivatives of region \( \mathbf{C}_{1,3} \) become equal to that of region \( \mathbf{C}_{1,4} \) and the function is differentiable at \( \alpha_{1,2}(t_1, t_2) \).
F  Appendix E

Property 7. If $T_1 \geq \text{var} \ T_2$, then

$$\min_{(t_1, t_2) \in \mathbb{R}^n \times \mathbb{R}^t} E_{F_1}[L(t_1, t_2, T)] \leq \min_{(t_1, t_2) \in \mathbb{R}^n \times \mathbb{R}^t} E_{F_2}[L(t_1, t_2, T)].$$

(347)

Proof. We successively consider the three strategies and the associated cost functions. We show that if the variability increases, each cost function increases. Apply Theorem 1, the definition of $t^*_j$, and the definition of optimal costs for regions $\mathbf{R}_1$ and $\mathbf{R}_2$ and the boundary in between the regions to see what happens when $T_1 \geq \text{var} \ T_2$.

We know that

$$E_P([t-T]^+) = \int_0^t F(T) dT,$$

(348)

$$E_P([T-t]^+) = \int_0^T F(T) dT + \mu - t.$$

(349)

The minimum of $L_1$. The optimal expected costs associated to probability distributions $F_1(\cdot)$ and $F_2(\cdot)$ can be rewritten as

$$E_{F_1}[L_1(t_1^*(F_i), t_2^*(F_i))] = -gt^*_1(F_i) - m_1t^*_1(F_i) - c_h t_j^2(F_i) + (m_1 - m_1') \int_0^{t_1^*(F_i)} F_1(T) dT + (m_2 + c_h) \int_0^{t_2^*(F_i)} F_1(T) dT,$$

and we thus have the following difference expression

$$E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_1(t_1^*(F_2), t_2^*(F_2))] = (m_1 - m_1') \int_0^{t_1^*(F_2)} (F_2(T) - F_1(T)) dT + (m_2 + c_h) \int_0^{t_2^*(F_2)} (F_2(T) - F_1(T)) dT.$$

As by optimality, one has

$$E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_1(t_1^*(F_1), t_2^*(F_1))] \geq E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_1(t_1^*(F_2), t_2^*(F_2))].$$

(350)

by (42), we conclude

$$E_{F_2}[L_1(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_1(t_1^*(F_1), t_2^*(F_1))] \geq 0.$$

(351)

The minimum of $L_b$. The optimal expected costs associated to probability distributions $F_1(\cdot)$ and $F_2(\cdot)$ can be rewritten as

$$E_{F_1}[L_b(t^*_1)] = -(g + m_1 + c_h) t_j^*(F_i) + (m_1 - m_1' + m_2 + c_h) \int_0^{t_1^*(F_i)} F_1(T) dT,$$
and we thus have the following difference expression

\[ E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_2^*)] = \sum m_2 - m_1^* + m_1 + c_{h,2} + g \int_0^{\tau_2^*(F_2)} (F_2(T) - F_1(T))dT. \]

As by optimality, one has \( E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_1^*)] \geq E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_2^*)] \), by (42), we conclude \( E_{F_2}[L_b(t_2^*)] - E_{F_1}[L_b(t_2^*)] \geq 0. \)

**The minimum of \( L_2 \)** The optimal expected costs associated to probability distributions \( F_1(\cdot) \) and \( F_2(\cdot) \) can be rewritten as

\[
E_{F_1}[L_2(t_1^*(F_1), t_2^*(F_1))] = -c_{h,2}t_2^*(F_1) - (m_1 + g)t_1^*(F_1) \\
+ \left( m_2 - m_1^* - s_1 + c_{h,2} \right) \int_0^{\tau_2^*(F_1)} F_1(T)dT \\
+ \left( m_1 + g + s_1 \right) \int_0^{\tau_1^*(F_1)} F_1(T)dT,
\]

and we thus have the following difference expression

\[
E_{F_1}[L_2(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_2(t_1^*(F_2), t_2^*(F_2))] = \left( m_2 - m_1^* - s_1 + c_{h,2} \right) \int_0^{\tau_2^*(F_2)} (F_2(T) - F_1(T))dT \\
+ \left( m_1 + g + s_1 \right) \int_0^{\tau_1^*(F_2)} (F_2(T) - F_1(T))dT.
\]

As by optimality, one has

\[
E_{F_2}[L_2(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_2(t_1^*(F_2), t_2^*(F_2))] \geq E_{F_1}[L_2(t_2^*(F_2), t_1^*(F_2))]. \tag{352}
\]

by (42), one concludes

\[
E_{F_2}[L_2(t_1^*(F_2), t_2^*(F_2))] - E_{F_1}[L_2(t_1^*(F_2), t_2^*(F_2))] \geq 0.
\]

Therefore, as the minimum in each region decreases with \( T_1 \geq T_2 \), then the global minimum which is the minimum of the minimum found in each region also decreases.