General Equilibrium Real and Nominal Interest Rates

by

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Abstract

We derive the general equilibrium short-term real and nominal interest rates in a monetary economy affected by technological and monetary shocks and where the price level dynamics is endogenous. Assuming fairly general processes for technology and money supply, we show that an inherent feature of our equilibrium is that any real variable dynamics, in particular that of the short-term real interest rate, is driven by both monetary and real factors. This money non-neutrality is generic, as it does not stem from any friction such as price stickiness, or from a particular utility function. Non-neutrality obtains because the ex ante cost of real money holdings is random due to inflation uncertainty. We then analyze in depth a specialized version of this economy in which the state variables follow square root processes, and the representative investor has a log separable utility function. The short-term nominal rate dynamics we obtain encompasses most of the dynamics present in the literature, from Vasicek and CIR to recent quadratic and, more generally, non-linear interest rate models. Moreover, our results pave the way to several new nominal term structures.
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1. Introduction

We propose a general equilibrium of a frictionless monetary economy in which money is an argument of the representative individual’s utility function. In a fairly general framework set in continuous time, we first derive and analyze the behavior of macroeconomic aggregates such as consumption, investment and real wealth and devote special attention to the inflation rate and the real and nominal interest rates. In a specialized version of the economy, where the representative agent has a log separable utility function and the state variables follow square root processes, we then provide explicit solutions to our model and derive in particular the implied dynamics for the real and short-term nominal rates. The main characteristic of our economy is that generically money is neither neutral nor supernormal, as monetary policy always affects the level and the dynamics of all real variables. The transmission mechanism works as follows. An individual holding real balances faces an opportunity cost that \textit{ex ante} is the nominal interest rate. However, the effective cost of money holding is not the nominal rate but the sum of the real rate and the inflation rate realized \textit{ex post}. Under uncertain inflation, the two costs are distinct since, at the beginning of each period, the first one is known while the second is random. Investors' real wealth, and thereby all other endogenous real variables, are affected by this uncertainty. To further investigate the consequences of money non-neutrality, we provide a closed form solution for a specialized economy that

\begin{footnotesize}
\footnote{According to the standard definition, money is supernormal with respect to a given variable if a change in its growth rate does not affect the \textit{level} of the variable [see Walsh (1998) page 56]. This paper deals essentially with supernormality, as it considers changes in the money growth rate, although it discusses neutrality occasionally. Since however we generally examine the impact of changes in the money growth rate on the dynamics of economic variables, not their levels, we will mostly make use of the following definition. Money is supernormal with respect to a given variable if a change in its growth rate does not affect the \textit{dynamics} (expectation and/or volatility of the growth rate) of the variable. It will be obvious from the context whether this definition or the standard one is used. Finally, the phrase “non-neutrality” will be used throughout to mean “non-supernormality”.}
\end{footnotesize}
can be viewed as the monetary extension of the real economy developed by Cox, Ingersoll and Ross (1985a, 1985b), hereafter CIR. Our monetary economy turns out to possess original properties as compared to pure real economies or monetary economies in which the real and the nominal sectors are linked artificially or not at all.

The first main contribution of this paper thus is to introduce a consistent framework of a monetary economy in which money is held because it provides utility and cannot in general be neutral in equilibrium, regardless of the shape of the utility function. Moreover, both expected and non-anticipated changes in the money supply rate of growth affect the level and the growth rate of all relevant variables. Such non-neutrality is achieved without introducing the imperfections (such as price stickiness and/or wage rigidities) characteristic of to-day standard models. The key is the correct modeling of the representative investor's wealth dynamics. This has important bearings on monetary theory, empirical analysis and interpretation of actual data, term structure modeling and asset pricing. As to the last topic, for instance, we show that Breeden’s (1979) consumption-based capital asset pricing model must be amended to incorporate an additional risk premium. The latter compensates investors for the inflation risk associated with their holdings of real balances and its magnitude depends on the value of the monetary policy parameters. In addition, although the structure of Merton’s (1973) intertemporal capital asset pricing model is preserved, each and every risk premium included in the expected real returns on financial assets is now affected by monetary policy.

The second main contribution concerns the behavior of the real interest rate in equilibrium. The short-term (in fact, continuous) real rate is equal to the expected return on real investment adjusted by a risk premium. The latter has two components, one related to consumption risk and the other to real balances risk. Since consumption and real money holdings are affected by monetary factors, so is the risk premium. This is the first transmission mechanism of monetary impulses to the real rate. Consumption risk has three components, technology risk, monetary risk, and the risk associated with
changes in the proportion of total wealth devoted to real investment. Since the latter risk is itself related to monetary risk, this compounds the impact of monetary policy on the real interest rate and makes its relationship to monetary factors highly non-linear. In the particular case of log separable utility functions, we solve completely for both the real rate level and its dynamics. They are shown to be very different from both what CIR obtained in their purely real economy and what is obtained in frictionless monetary economies in which money is in fact neutral (for a representative example, see Bakshi and Chen (1996)). To the best of our knowledge, such results for the short-term real rate are novel.

Another main contribution concerns the behavior of the equilibrium nominal rate of interest. We recover the well-known result that this rate is equal to the marginal rate of substitution between real money balances and consumption. Although it is in general affected by both technological and monetary parameters, it is solely influenced by monetary factors in the log utility case. Its dynamics does nevertheless encompass most interest rate models offered in the literature, which therefore obtain as special cases of ours. First and foremost, we can recover CIR’s square root interest rate model with the crucial provision that the latter was derived for the real rate, not the nominal rate as here. This provides a sound theoretical background to the numerous papers that used CIR’s model as if it was obtained in a monetary economy and vindicates its adoption as the nominal interest rate model of a truly monetary economy in general equilibrium. Second, not only well known affine models of the term structure but also more complicated ones such as the non-linear models of Ahn and Gao (1999) or the log normal model of Miltersen, Sandman and Sonderman (1997) can be derived as particular cases. Third, our model also embeds the quadratic term structure model recently developed by Leippold and Wu (2000) and Ahn, Dittmar and Gallant (2002) and empirically tested with some success by Brandt and Yaron (2001).

Our results also contribute to the hotly debated issue of the presence of non-linearities in the short-term nominal rate dynamics. No consensus among academics and professionals
has been reached or is even foreseeable [see Chapman and Pearson (2001) for a review of the main issues]. Using non-parametric methods, recent research that includes Ait-Sahalia (1996), Stanton (1997), Conley et al. (1997), and Boudoukh et al. (1999) in a multivariate case, showed that the drift of the nominal rate process is a non-linear function of the rate level. Those results constituted a direct challenge to the standard affine models (such as CIR’s) that are generally used in parametric investigations of bond prices. Following this line of criticism, some authors such as Ahn and Gao (1999) proposed non-linear parametric models of the term structure and provided explicit solutions for bond prices. On the other hand, other authors highly questioned the results of the non-parametric tests. For instance, Chapman and Pearson (2000) and Pritsker (1998) showed that non-parametric tests might reject even true models of the interest rate dynamics. An important theoretical support (or lack of) to any parametric model of the term structure is whether it is grounded on explicit equilibrium considerations. Our results will allow one to infer under what conditions the dynamics of the short-term nominal (and real) rates will exhibit a non-linearity in the drift and a non-exponential relationship between the diffusion parameter and the rate level. Regarding the nominal rate, (non-) linearity is a direct consequence of a more fundamental (non-) linearity: it depends on whether the drift and diffusion parameters of the money supply growth rate are linear or not in the relevant state variable(s). This is not the case for the real rate. Because money is not neutral, the dynamics of the short rate will always exhibit non-linearities even though the parameters of the technological and money supply processes are linear functions of the state variables. Therefore, in contrast with some authors (Ang and Bekaert (2002) for example) who have to invoke the presence of (most often unidentified) exogenous shocks, such as regime switching, to explain interest rate non-linearities, we show why and how the latter are related to endogenous factors.

The real and nominal interest rates are linked by the expected inflation rate and the inflation risk premium. In both the general and specialized versions of our economy, the endogenously derived parameters of the price level dynamics are highly non-linear functions of the parameters of the money supply process. Since the inflation risk
premium (as indeed the risk premium of any risky asset) is also influenced by the monetary policy parameters, the impact of money on the dynamics of the real interest rate is compounded. This also highlights the limits of many empirical investigations that attempted to extract the inflation risk premium from bond market prices only. A methodology such as VAR (Vector Auto-Regression) where all the relevant real and nominal processes are jointly determined in an endogenous manner is advocated to ensure estimation consistency. This of course requires consistent assumptions regarding the identity (and identification) of relevant processes. For instance, the impact of the systematic (as opposed to the unexpected) component of monetary policy is generally ignored, although Cochrane (1998) has shown (in a different context) that adding this component enhances the significance of results. And consistency is best achieved when empirical testing is grounded on results of a general equilibrium framework.

Finally, our model has obvious implications as to the potential factors explaining time series and cross sectional features of nominal bond prices. In general, the factors found in the literature are related to the properties of the term structure itself such as the general level, steepness and convexity of the curve, or the volatility of the interest rates. This is similar to explaining the cross section of asset returns by the return on the market portfolio, its volatility, skewness and kurtosis, and/or by the returns of particular “ad hoc” portfolios deemed to reflect common exposures to (generally) non-specified risks2. Thus, bond returns are not related to fundamental economic risks. Recently, some empirical effort was devoted to this issue. Ang and Piazzessi’s (2002) investigation showed the relevance of macroeconomic variables for explaining bond prices and their superiority as compared to non-observable, endogenous, factors. Our paper identifies exactly the factors affecting the real and the short-term nominal rates. While three factors are needed to explain the level and dynamics of the real rate, namely the technology, the money supply and the investment/wealth ratio, one factor only, the

2 Liew and Vassalou (2000) using data from ten countries provide preliminary empirical evidence that, these portfolios may be linked to the future growth in the Gross Domestic Product.
money stock, plays a role for the determination of the nominal rate. Our identification thus is parsimonious and provides theoretical support to the many papers that showed that one factor in general explains about 90% of nominal bond price fluctuations (see for instance Chapman and Pearson (2001)). Moreover, estimating the level and dynamics of the short-term nominal rate does not require the use of consumption data.

The remainder of the article is organized as follows. Section 2 reviews the relevant literature and its relation to our work. Section 3 presents the monetary economy under investigation, assets and goods that are traded, the economic agents’ behavior and the general stochastic processes that drive the real returns on the available technology and the nominal money supply. Section 4 derives the equilibrium in the general economy and characterizes the real wealth growth rate, the inflation rate, the demand for risky assets, the expected real excess returns on financial claims, the real and nominal rates of interest, and the inflation risk premium. Section 5 derives explicit solutions for all endogenous processes and variables in a specialized version of our economy in which the state variables follow square root processes, and the representative investor has a log separable utility function. Special attention is devoted to the dynamic behavior of the real and nominal interest rates. Section 6 concludes. A mathematical appendix gathers all proofs and technical derivations.
2. Related Literature

The abundant and ever growing literature on term structure modeling and interest rate derivatives pricing witnesses the sizeable progress that has been accomplished in recent years both at the theoretical and the empirical levels. The adoption of new non-parametric techniques as well as parametric techniques to estimate the term structure enhanced our understanding of the behavior of bond market prices and of the shortcomings of standard models³. For instance, several papers provided a thorough analysis of the implications and limitations of affine term structure models a la CIR⁴. Others proposed alternatives to the multifactor generalization of CIR such as the extension of Heath, Jarrow, and Morton’s (1992) basic framework to the existence of an infinite number of factors⁵. All these models have also been calibrated and implemented to evaluate various interest rate derivatives.

By comparison, very little effort has been devoted, somewhat surprisingly, to providing these new models a sound economic background. The most widely used approach simply consists in assuming on a priori grounds a given dynamics for the short-term nominal rate and then deriving the dynamics of bond prices and/or the price of derivatives. Although the CIR model of the term structure was set in a general equilibrium framework, this was the case for almost none of the extensions proposed thereafter. In addition, the CIR model has been derived for a real, non-monetary, economy and nonetheless was used as if it were applicable to a nominal term structure.

⁵ See Santa-Clark and Sornette (2001) and Goldstein (200), and the references therein.
Others have unfortunately followed this tradition. For example, Longstaff and Schwartz (1992) and Ahn, Dittmar and Gallant (2002) derive their so-called nominal term structure dynamics within a CIR-like purely real economy. More generally, Jin and Glasserman (2001) show that every Heath, Jarrow, and Morton (1992) arbitrage free model of the term structure can be supported by a real-economy equilibrium a la CIR. This can be true only if interest rates are interpreted as real ones, or, equivalently, if the inflation rate is deterministic.

This paper, however, is not the first one to attempt to build a truly monetary economy, and to derive the equilibrium real and nominal term structures. The standard approach, followed by CIR themselves and others, merely consists in adding to CIR’s framework an *exogenous* process for the price level or the inflation rate and then deriving the relevant variables, assuming along the way that money has no real effect. Other authors such as Pennacchi (1991) added artificially some non-neutrality, for instance by assuming on a priori grounds that the drift of the technological process depends on inflation in an otherwise CIR economy, the dynamics of the inflation rate being exogenous.

Important progress has however been accomplished by Bakshi and Chen (1996) for a domestic economy and Basak and Gallemeeyer (1999) for an international economy. These authors built monetary economies in which the price level is found endogenously within a money-in-utility framework with a representative agent. Unfortunately, both papers proposed a partial equilibrium framework in which output and consumption are in fact exogenous. Not surprisingly then, these models exhibit money superneutrality in equilibrium. Therefore, the present model of a truly monetary economy that leads to money non-neutrality without « ad hoc » assumptions fills an obvious gap. In particular,

6 Buraschi and Jiltsov (2002) extend Bakshi and Chen’s (1996) analysis to a production economy with taxes. Introducing a capital depreciation by firms that is imperfectly indexed on inflation does yield money non-neutrality. However, once this (possibly important) friction is removed, money does not affect the dynamics of physical capital any more in this model and recovers its neutrality.
our dynamics for the real and nominal interest rates are dramatically different from each
other, and much more general than the ones derived in the last quoted papers.

Our monetary model differs from existing economic models in other ways. First, time is
continuous rather than discrete. Such an assumption generates tractable solutions for all
endogenous variables that possess intuitive economic interpretations and are amenable
to direct empirical investigations. In addition, the dynamics of all variables are obtained
without approximation while, in discrete time analysis, linear approximations around the
steady state must usually be performed\(^7\). Second, the technological returns and the
nominal money growth are assumed to follow fairly general stochastic processes. Third,
the output growth rate, the rate of inflation, the expected real excess returns on the
technology and on pure financial assets, the real and nominal interest rates and the
inflation risk premium are jointly determined in the general case. Additional closed form
solutions are derived for the price level, the levels of output, investment, and
consumption, and the optimal proportions of wealth invested in the technology and in
real balances in a special case. Even more importantly, we derive the precise dynamics
followed by the real and nominal short-term rates. The short-term nominal rate dynamics
we obtain encompasses most of the dynamics that have been proposed in the literature,
including highly non-linear ones, as special cases. Fourth, individuals can trade on pure
financial (contingent) claims in addition to the real and nominal savings accounts and to
(equity) claims to technological returns, which broadens the scope of the analysis.

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\(^7\) See Walsh (1998) for an exposition of discrete time steady states and short-term deviations about them
in various monetary economies.
3. The economic framework

The structure of the real sector of the economy is similar to that of CIR. We add a monetary sector by introducing money supply on the part of the Central Bank and money demand on the part of investors. The representative individual’s wealth dynamics then is derived.

3.1 Real sector, traded assets and money supply process

In the considered economy, there is a single physical good that may be allocated to either consumption or investment. When variables are said to be expressed in real (respectively, nominal) terms, it is understood that the implicit numeraire used is this physical good (respectively, money). The good is produced by a single technology (firm)\(^8\). The amount (real value) of the good invested at date \(t\) in the technology is denoted by \(\eta(t)\). Production over time through the technology is governed by the following stochastic differential equation (SDE):

\[
d \eta(t) = \eta(t) \mu_{\eta}(t, Y(t)) dt + \eta(t) \sigma_{\eta}(t, Y(t))' dZ(t)
\]

where \(Z(t)\) is an \((N + K) \times 1\) dimensional Wiener process in \(\mathbb{R}^{N+K}\), \(Y(t)\) is a \(K \times 1\) dimensional vector of state variables, \(\mu_{\eta}(t, Y(t))\) is a bounded function of \(t\) and \(Y\), and \(\sigma_{\eta}(t, Y(t))\) is a bounded \((N + K) \times 1\) vector valued function of \(t\) and \(Y\). The Wiener process is defined on the usual complete probability space \((\Omega, F, P)\) where \(P\) is the true (historical) probability. We impose the normalization \(\eta(0) = 1\). Since we make the classic simplifying assumption that the consumption/investment good is a non-durable

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\(^8\) The generalization to \(N\) different technologies, as done in Lioui and Poncet (2001), although easy, makes the notation heavier and adds no economic intuition.
good and that investment in the technological process is continuously destroyed, there is strictly speaking no capital accumulation.

The dynamics of the K state variables are determined by the following system of SDEs:

\[
dY(t) = \mu_Y(t, Y(t))dt + \Sigma_Y(t, Y(t))dZ(t)
\]

where \( \mu_Y(t, Y(t)) \) is a bounded \( K \times 1 \) vector valued function of \( t \) and \( Y \) and \( \Sigma_Y(t, Y(t)) \) is a bounded \( K \times (N + K) \) matrix valued function of \( t \) and \( Y \).

Consider an investor in this economy. At time \( t \), he or she holds a number \( A(t) \) of units of the technology, whose real value is \( A(t)\eta(t) \). The proportion of wealth invested in the technology is denoted by \( \alpha(t) \equiv \frac{A(t)\eta(t)}{w(t)} \), where \( w(t) \) is the investor’s real wealth.

In the remainder of the paper, the terms “technology”, “equity”, and “real investment” will be used interchangeably.

In addition to real investment in the technology, investors have access to various markets for contingent (financial) claims to units of the consumption good. We assume that there are \( H \) non-redundant contingent claims. All of them are spot assets\(^9\). These contingent claims are in zero net supply and their real prices are assumed to follow:

\[
ds(t, Y(t)) = I_s(t, Y(t))\mu_s(t, Y(t))dt + I_s(t, Y(t))\Sigma_s(t, Y(t))dZ(t)
\]

\(^9\) As shown in Lioui and Poncet (2002), when some contingent claims are not traded on a spot basis but on a forward basis, the technical derivation is significantly more complicated.
where $I_s(.)$ is a diagonal matrix the $i^{th}$ diagonal term of which is $s_i$, and $\mu_s(t, Y(t))$ and $\Sigma_s(t, Y(t))$ are general processes to be found *endogenously*.

In addition to equities and contingent claims, individuals have access to two money market accounts, both of which are in zero net supply\(^{10}\). The real (index) savings account is denominated in the consumption/investment good and has an instantaneous return that is riskless in real terms and equal to the real interest rate $r(t)$. The nominal savings account is denominated in dollars and is riskless in nominal terms, its instantaneous yield being the nominal interest rate $R(t)$.

Depending on the magnitude of $H$, the number of non-redundant contingent claims, the financial market is complete or incomplete, at will. We stress that *none* of our results depends on whether the financial market is complete or not\(^{11}\). Note that if $H = (K+N-2)$ then investors are able to trade $(H + 2) = (K + N)$ non-redundant assets with $(N + K)$ sources of uncertainty driving the economy. However, even in this case, the market still is incomplete since investment in the technology is subject to a short sale constraint.

Lastly, the Central Bank issues money and arbitrarily sets its nominal rate of return to zero. The reason why money is not strictly dominated by the nominally riskless money market account yielding $R(t)$ is that it helps reducing (implicit) transaction costs. It is therefore desired for the liquidity services it provides. The money supply process is exogenous to the model. The outstanding nominal quantity of money is not a state

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\(^{10}\) Real (index) long-term bonds are not explicitly introduced into the analysis, as they are special cases of financial claims. Nor are nominal bonds, since they are claims to $1/p(t)$, rather than one, units of the consumption good, where $p(t)$ is the general price level at the bond maturity date $t$.

\(^{11}\) There is a minor difference in the assumption regarding individuals between the two cases. If the market is complete, we can assume that there is a representative individual. If not, the existence of a representative agent is more problematic and in general we have to resolve to the stronger assumption that individuals are identical. From now on, we will use the phrase “representative individual”, even though the market is incomplete.
variable itself, but may be influenced by the state variables to which the Central Bank presumably reacts. To allow for this possible dependence, its dynamics is expressed as:

\[ dM(t) = M(t)\mu_M(t, Y(t))dt + M(t)\sigma_M(t, Y(t))dZ(t) \] (4)

where \( \mu_M(t, Y(t)) \) is a bounded function of \( t \) and \( Y \) and \( \sigma_M(t, Y(t)) \) is a bounded \((N + K) \times 1\) vector valued function of \( t \) and \( Y \). These two functions are policy parameters and thus are \textit{exogenous} to the model. The economic quantities to be derived endogenously will be, in particular, functions of \( \mu_M(.) \) and \( \sigma_M(.) \). Note however that the parameters of the production process, \( \eta_{\mu}(t, Y(t)) \) and \( \eta_{\sigma}(t, Y(t)) \), which depend on the state variables, do not depend directly on \( \mu_M(t, Y(t)) \) and \( \sigma_M(t, Y(t)) \). The initial stock of money \( M(0) \) is compatible with the initial price level \( p(0) \) but otherwise arbitrary.

Monetary policy has two components: the systematic part is reflected in the instantaneous drift \( \mu_M(.) \) while the discretionary, unexpected, part is embedded in the instantaneous variance \( \sigma_M(.) \). Under rational expectations, the systematic part is perfectly anticipated by economic agents. Monetary policy will be shown to influence real economic variables through both this systematic component \( \mu_M(.) \) and the surprise element \( \sigma_M(.) \).

The general price level, i.e. the money price of one unit of the consumption-investment good, \( p(t) \), is assumed to obey the following SDE:

\[ dp(t) = p(t)\mu_p(t, Y(t))dt + p(t)\sigma_p(t, Y(t))dZ \] (5)

where \( \mu_p(t, Y(t)) \) and \( \sigma_p(t, Y(t)) \), the latter an \((N + K) \times 1\) vector valued function, are to be found \textit{endogenously} as part of the solution to the equilibrium problem.
Finally, trading in all financial and monetary assets and in the technology takes place continuously in frictionless and arbitrage-free markets and at equilibrium prices only.

### 3.2 Preferences and the budget constraint

As stated in the introduction, we use the money-in-the-utility-function model originally developed by Sidrauski (1967). Thus, the infinitely-lived representative investor in this economy maximizes the expected utility of her intertemporal consumption \( c(t) \) and real money balances holdings \( m(t) \) under her budget constraint. Therefore, her consumption and portfolio decisions maximize:

\[
\mathbb{E}\left[ \int_1^{\infty} U(s, c(s), m(s))ds \right] 
\]

where \( U \) is assumed to be a twice continuously differentiable, increasing and strictly concave utility function\(^{12}\), \( \mathbb{E} \) is the expectation operator conditional on current endowments and the state of the economy. Real money balances \( m(t) \) are equal to \( M(t)/p(t) \). When maximizing (6), the representative investor is assumed to limit her attention to admissible controls only.

From now on, we delete the explicit dependence of the variables on time and the state variables, unless stated otherwise. The budget constraint then reads:

\[
dw = w\alpha \frac{d\eta}{\eta} + w\eta^I_s^{-1}ds + w\delta\left( Rd\frac{dp^{-1}}{p^{-1}} \right) + w\varphi dt - cdt + \frac{m}{p^{-1}} 
\]

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\(^{12}\) This implies the following classic conditions, where subscripts on \( U \) denote partial derivatives: \( U_c > 0, U_m > 0, U_{cc} < 0, U_{mm} < 0, U_{cm} < 0 \) and \( U_cU_{mm} - (U_{cm})^2 > 0 \).
where \( \theta \) is the vector of proportions of real wealth invested in the contingent claims, \( \delta \) (respectively, \( \phi \)) is the proportion of real wealth invested in the nominal money market account (respectively, the real money market account) and \( c \) is the consumption rate. The last term is the opportunity cost of holding money, given by the change in the real price of one unit of currency.

The wealth dynamics (7) deserves the two following comments. First, the \textit{ex post real} return on the nominal money market account is equal to \( R(t)dt \), the nominal interest rate, plus the realized rate of depreciation of the purchasing power of money from \( t \) to \( t + dt \), \( \left( \frac{dp^{-1}}{p^{-1}} \right) \). Second, the direct \textit{ex post} cost of holding real balances \( m(t) \) between date \( t \) and date \( t + dt \) is, similarly, proportional to the decrease in the purchasing power of money. Thus, the exact \textit{ex post} opportunity cost of holding one unit of real cash balances is \textit{not} the nominal interest rate \( R(t)dt \), but the sum \( \left[ r(t)dt - \frac{dp^{-1}}{p^{-1}} \right] \). In a world of certainty, the latter sum would of course be equal to \( R(t)dt \). However, under uncertain inflation, these two quantities differ. In particular \( R(t)dt \) is deterministic while \( \left[ rdt - \frac{dp^{-1}}{p^{-1}} \right] \) is stochastic. This difference will prove crucial for the non-neutrality of money.

Using \( \phi = 1 - \delta - \alpha - \theta'1_H - m/w \), the wealth dynamics (7) can be rewritten as:

\[
\begin{align*}
dw = w\alpha \left[ \frac{dn}{\eta} - rdt \right] + w\theta' \left[ I_s^{-1} ds - 1_H rdt \right] + w\delta \left[ Rdt + \frac{dp^{-1}}{p^{-1}} - rdt \right] \\
+ wrdt - cdt - m \left[ rdt - \frac{dp^{-1}}{p^{-1}} \right] 
\end{align*}
\]

(8)
Equation (8) clearly demonstrates the need for a correct specification of the ex post opportunity cost of real money holdings. If \( R(t)dt \) were (wrongly) assimilated to 
\[
\left[ r(t)dt - \frac{dp^{-1}}{p^{-1}} \right],
\]
the third term in brackets on the RHS of the equation would vanish and, more importantly, the last term \( m \) would be multiplied by \( Rdt \), a deterministic term. Most results would be then altered significantly.

Now, using (1), (3) and (5) yields the representative agent’s wealth dynamics:

\[
dw = \left[ w\alpha[\mu_{\eta} - r] + w\theta'[\mu_s - 1_{H} r] + w\delta[R - r - \mu_p + \sigma_p \sigma_p'] \right]
\]
\[
+ \left[ wr - c - m[\mu_p - \sigma_p \sigma_p'] \right]
\]
\[
+ \left[ w\alpha\sigma_{\eta}' + w\theta'\Sigma_s - (\delta w + m)\sigma_p' \right]dZ
\]

We can now derive the general equilibrium in our monetary economy.
4. The General Equilibrium

At this level of model generality, completely closed-form solutions for the economic general equilibrium cannot be obtained, but major insights as to the transmission of monetary impulses to real and nominal variables and as to the equilibrium behavior of interest rates and asset prices can nevertheless be gained. In the remainder of this section, we drop the explicit dependence of the variables on time and the state variables to ease the exposition.

We start by showing explicitly how the transmission mechanism of monetary impulses works, then derive the real and nominal rates of interest and finally study the implications of our model for asset pricing in general and the yield curve in particular.

4.1. The transmission mechanism

For the economy to be in equilibrium, the following market clearing conditions must be satisfied:

(i): total wealth, taken without loss of generality to be that of the representative individual, must be equal to the total amount invested in the technology plus the real value of money balances held, i.e. $w\alpha + m = w$,

(ii): net holdings in each of the two money market accounts and in each of the various contingent claims must be equal to zero, i.e. $\delta = 0$, $\varphi = 0$ and $\theta = \theta$\textsuperscript{13}, and

(iii): money supply must equal money demand, i.e. $\frac{M}{p} = m = w(1 - \alpha)$.

\textsuperscript{13} Note that one of these conditions is redundant when the others (along with condition (i)) are satisfied.
Under these conditions, we can derive the following

**Proposition 1 (real wealth):**

*In equilibrium, the aggregate real wealth evolves over time as\(^1\):*

\[
\frac{dw}{w} = \mu_w dt + \sigma_w dZ
\]

(10)

where its expected instantaneous growth rate \(\mu_w\) is equal to:

\[
\mu_w = \frac{1 - \alpha}{\alpha} \mu_M + \frac{1 - \alpha}{\alpha^2} \sigma_M \sigma_M + \mu_\eta - \frac{c}{\alpha w} + \left( -\mu_\alpha + \sigma_\alpha \right) \\
-\sigma_\alpha \sigma_\eta + \frac{2 - \alpha}{\alpha} \sigma_\alpha \sigma_M - \frac{1 - \alpha}{\alpha} \sigma_\eta \sigma_M + \frac{1 - \alpha}{\alpha^2} \sigma_M \sigma_M
\]

(11)

and its instantaneous volatility \(\sigma_w\) is equal to:

\[
\sigma_w = \sigma_\eta - \frac{1 - \alpha}{\alpha} \sigma_M - \sigma_\alpha
\]

(12)

where \(\mu_\alpha\) and \(\sigma_\alpha\) are the drift and the volatility, respectively, of the *relative changes* in the proportion of wealth devoted to real investment.

Results (11) and (12) are derived from the market clearing conditions and the representative agent’s wealth constraint only. First order conditions for an optimum are not used at all at this point. An immediate consequence is that those results hold *regardless of the shape of the representative agent’s utility function*. In particular, they do not depend on consumption and money being separable or not in the investor’s utility function. The fact that monetary parameters matter in (11) and (12) thus is inherent in a

\(^1\) Recall that all variables depend on time and the state variables, dependence omitted for easier readability.
true monetary economy. \textit{Money cannot be (neutral or) superneutral in general} and both the systematic ($\mu_M$) and unexpected ($\sigma_M$) components of the money supply will affect the wealth dynamics.

According to equation (11), an increase in the average growth rate of the money supply induces a decrease in the expected growth rate of real wealth, since the larger are money balances held at equilibrium, the higher is their opportunity cost. Inflation indeed is a tax on real money holdings. Moreover, real wealth average growth increases with the variance of the money supply process $\sigma_M^2$, because, as returns on real money holdings become more volatile, the real demand for money declines and real investment rises on average. It also increases, as anticipated, with the expected return on real investment $\mu_\eta$ and decreases with the consumption to investment ratio ($c/\alpha w$).

It is also a positive function of $\left( -\mu_a + \sigma_a' \sigma_a \right)$, the drift of $(1/\alpha)^{15}$. If the investment/wealth ratio $\alpha$ is expected to decrease, $(1/\alpha)$ is expected to increase and this will cause an increase in the expected growth rate of real wealth. Indeed, an expected decrease in $\alpha$ implies an increase in the relative (to wealth) demand for real balances, which lowers the expected inflation rate$^{16}$. Since the expected opportunity cost of money holdings falls, the expected growth of real wealth rises.

It is useful to remark that in CIR’s real, moneyless, economy, $\alpha$ is by construction the constant one, which simplifies the analysis but fails to take into account the volatility of this ratio empirically observed in actual economies. Our analysis is richer and will be

\begin{itemize}
  \item[	extsuperscript{15}] The variance term $\sigma_a' \sigma_a$ is due to Itô’s calculus and thus has no precise economic interpretation. Therefore, instead of using $\mu_a$, the drift of the change rate in the investment/wealth ratio, we prefer to ground our reasoning on $(-\mu_a + \sigma_a' \sigma_a)$, the drift of $1/\alpha$.
  \item[	extsuperscript{16}] See the second term on the right hand side of equation (14) below.
\end{itemize}
shown below to have important implications as to the behavior of major variables such as investment, consumption and asset prices\(^{17}\).

As to the additional terms in equation (11), they reflect the second-order impact of the relevant covariances between technological returns, money growth and capital investment.

As evidenced by equation (12), the volatility of the wealth growth rate depends on both real and monetary uncertainty. Since this volatility involves vectors, not scalars, it is in fact easier to comment on the variance, given by:

\[ \sigma^2_w = \sigma^2_\eta + \left( \frac{1-\alpha}{\alpha} \right)^2 \sigma^2_M + \sigma^2_\alpha - 2 \frac{1-\alpha}{\alpha} \sigma_\eta \sigma_M - 2 \sigma_\eta \sigma_\alpha + 2 \frac{1-\alpha}{\alpha} \sigma_M \sigma_\alpha \] (12')

As intuition suggests, this variance increases with that of returns from investment in the technology and that of the change rate in the optimal investment/wealth ratio. The net effect of monetary uncertainty on the variance of real wealth is indeterminate as it depends positively on the variance of the money supply but also on the covariance between monetary and real shocks. If this covariance is positive, monetary uncertainty may reduce the variance of real wealth. If not, it unambiguously increases the latter variance.

Since the dynamics of real wealth is affected by monetary policy, so is that of real investment, since the latter is equal to \((\alpha w)\). Moreover, since our economic system is Markovian, optimal consumption can be written in feedback form \( c = c(t, w, Y) \). For a given wealth, the state of the economy, which depends on the state variables, does in general influence the optimal controls, namely consumption and money holdings. It

\(^{17}\) Addressing a different but related issue, Lettau and Ludvingson (2001a, 2001b) report that the consumption/wealth ratio is a significant factor predicting the dynamics of financial asset returns. Hahn and Lee (2001), however, document that the relationship is highly instable over time, which therefore deteriorates the predicting power of the ratio.
immediately follows that the dynamics of optimal consumption depends on both real and monetary parameters, and in particular on the systematic component of the money supply process. For instance, the volatility of consumption is expressed as:

\[
\sigma_c = \frac{1}{c} \left( \sigma_w c_w + \sum c_Y \right) = \left( \sigma_c - \frac{1-\alpha}{\alpha} \sigma_M - \sigma_\alpha \right) \frac{w}{c} c_w + \frac{1}{c} \sum c_Y \]

(13)

where \( c_w \) and \( c_Y \) denote partial derivatives with respect to wealth and the state variables, respectively.

As to the behavior of the general equilibrium inflation rate, it is given in the following

**Proposition 2 (inflation):**

In equilibrium, the expected instantaneous rate of inflation is equal to:

\[
\mu_p = \frac{1}{\alpha} \mu_M + \left( -\mu_\eta + \sigma_\eta \right) + \frac{c}{\alpha w} - \frac{1}{1-\alpha} \left( -\mu_\alpha + \sigma_\alpha \sigma_\alpha \right) \\
+ \frac{1}{1-\alpha} \sigma_\alpha \sigma_M + \frac{1}{(1-\alpha)^2} \sigma_\alpha \sigma_\alpha - \frac{1}{\alpha} \sigma_\eta \sigma_\alpha - \frac{1}{\alpha} \sigma_\alpha \sigma_\eta
\]

(14)

and its instantaneous volatility is given by:

\[
\sigma_p = \frac{1}{\alpha} \sigma_M - \sigma_\eta + \frac{1}{1-\alpha} \sigma_\alpha
\]

(15)

Again, results (14) and (15) are derived from the market clearing conditions and the representative agent’s wealth constraint only. The expected rate of inflation given by equation (14) depends in a complex way on real and monetary factors. In addition to being positively related to the average growth rate of money \( \mu_M \), expected inflation is a decreasing function of the average productivity of real investment. It is a positive function of the consumption/investment ratio \( (c/\alpha w) \) since investment and expected output decrease when consumption increases. It is also decreasing in the expected
change in $1/\alpha$, namely $(-\mu_\alpha + \sigma_\alpha ' / \sigma_\alpha)$. Indeed, if the proportion of wealth devoted to real investment is expected to increase, $1/\alpha$ is expected to decrease, and this will cause an increase in expected inflation since the relative demand for money is expected to decrease. The additional terms in equation (14) reflect the second-order impact of the relevant covariances between technological returns, money growth and capital investment.

The relationship between inflation volatility and money growth instability is not one-to-one. From equation (15), we have:

$$
\sigma_p ' \sigma_p = \frac{1}{\alpha^2} \sigma_M ' \sigma_M + \sigma_\eta ' \sigma_\eta + \frac{1}{(1-\alpha)^2} \sigma_\alpha ' \sigma_\alpha + 2 \frac{1}{\alpha} \frac{1}{1-\alpha} \sigma_M ' \sigma_\alpha \\
-2 \frac{1}{\alpha} \sigma_M ' \sigma_\eta - 2 \frac{1}{1-\alpha} \sigma_\eta ' \sigma_\alpha
$$

(15')

The variance of the inflation rate depends positively on the instability of the money growth rate and negatively or positively on that of real investment returns, depending on the sign of the covariance between monetary and real shocks. Moreover, if the demand for real money balances decreases ($\alpha$ becomes larger), $\sigma_p$ decreases in general because of a smaller impact of monetary uncertainty.

Since real balances are part of real wealth, the optimal allocation of wealth between financial assets will differ from that obtained in a money-less economy. The influence of money on the representative investor’s optimal allocation is given in the following
Proposition 3 (demand for risky assets):

The equilibrium demand for risky assets is equal to:

\[
\begin{pmatrix}
\alpha \\
\theta \\
\delta
\end{pmatrix} = \left(\Sigma \Sigma'\right)^{-1} \begin{pmatrix}
\mu_n - r \\
\mu_s - \mathbf{1}_\mu \bar{r} \\
R - r - \mu_p + \sigma_p' \sigma_p
\end{pmatrix} - \frac{J_w}{wJ_{ww}} + \left(\Sigma \Sigma'\right)^{-1} \Sigma \Sigma_Y' - \frac{J_w Y}{wJ_{ww}} + \frac{m}{w} \left(\Sigma \Sigma'\right)^{-1} \Sigma \sigma_p \tag{16}
\]

where:

\[
\Sigma = \begin{pmatrix}
\sigma_n' \\
\Sigma_s' \\
- \sigma_p'
\end{pmatrix}
\tag{17}
\]

The optimal dynamic trading strategy has three components. The first term on the RHS of (16) is the usual mean-variance (myopic) speculative component while the second term is the traditional vector of Merton-Breeden hedges against the fluctuations of each and every state variable. The last component is original and due to the presence of money. It is a minimum variance hedge ratio whose purpose is to hedge real balances against inflation risk. As is well known from the abundant literature on optimal hedging, when an expected utility maximizer determines her dynamic strategy while being constrained to hold a given position in an asset, the optimal strategy will contain a minimum variance hedge ratio to cover this position. Similarly, individuals are bound to hold real balances for the services it provides, which entails in essence a constrained position and the result immediately follows. This implies that the existence of an asset perfectly correlated with the inflation rate is desirable (Pareto improving) since it would provide a perfect hedge against the risk stemming from real money holdings. This result can thus be seen as the theoretical support for the development of indexed bond markets and CPI derivatives. Campbell, Chan and Viceira (2002) provide evidence that a
conservative investor should hold large positions on indexed bonds when they are available for trade, as this strategy greatly increases her expected utility.

Now, Merton-Breeden hedging components against the K state variables are known to lead to asset returns that contain K risk premiums at equilibrium in addition to the premium on the market portfolio. Unlike these components, the last term on the RHS of (16) does not depend on the utility parameter \((-J_{WY}/J_{WW})\), although it is not strictly speaking preference free since optimal money holdings \((m)\) and wealth \((w)\) depend on the investor’s utility. This contrasts with Brennan and Xia (2001) who address an investor’s dynamic asset allocation issue under uncertain inflation when only nominal assets can be traded. Not surprisingly, their investor protects his wealth against inflation risk and his portfolio, unlike here, exhibits a pure Merton-Breeden, utility dependent, hedging term\(^{18}\). Nevertheless, our third term obviously does not cancel out in equilibrium and thus commands a risk premium for each and every asset. In view of the current hotly debated issue on asset return predictability, the impact of this component on asset returns is worth studying. Since aggregate consumption is influenced by money growth, we can already assert that the risk premium in Breeden’s consumption-based CAPM for real asset returns will also be affected. This is the channel through which monetary shocks are transmitted to equilibrium asset returns. Within the approach of Merton’s intertemporal CAPM, the impact of this hedge component on the asset risk premiums is quite involved and will be analyzed thoroughly in the following sub-sections. We turn first to the determination of the equilibrium short-term interest rates.

\[^{18}\text{The authors do not consider equilibrium issues. It is clear however that in such a (partial) framework inflation risk will be priced in equilibrium as any other Merton-Breeden state variable.}\]
4.2. Real and nominal short term rates

Since money is not neutral, the parameters of the money supply process affect the equilibrium real short-term rate, as shown in the following

**Proposition 4 (real interest rate):**

*In equilibrium, the instantaneous real rate of interest is equal to:*

\[
r = \mu_\eta - \left( -\frac{cU_{cc}}{U_c} \right) \sigma_{\eta} \sigma_{c} - \left( -\frac{mU_{cm}}{U_c} \right) \sigma_{\eta} \sigma_{m}
\]

(18)

*or, equivalently, to:*

\[
r = \mu_\eta - \left( -\frac{cU_{cc}}{U_c} \right) \frac{w}{c} \sigma_{c} + \left( -\frac{mU_{cm}}{U_c} \right) 1 - \frac{\alpha}{\sigma_{\eta} \sigma_{M}}
\]

\[
+ \left( -\frac{cU_{cc}}{U_c} \right) \frac{w}{c} \sigma_{c} + \left( -\frac{mU_{cm}}{U_c} \right) \frac{1}{1-\alpha} \sigma_{\eta} \sigma_{u}
\]

\[
+ \left( -\frac{cU_{cc}}{U_c} \right) \frac{1}{c} \sigma_{\eta} \sum \sigma_{c_Y}
\]

(19)

Money non-neutrality is apparent in equation (18). To the best of our knowledge, this is the first time this expression is derived for the short-term real rate within a monetary economy in which money is not neutral. It is worth noting that non-neutrality is not solely related to the second risk premium on the RHS of the equation associated with real balances. This is because aggregate consumption, which appears in the first risk premium, also is affected by monetary parameters. Thus, the non-neutrality result remains valid if the representative investor's utility function is separable in its two
arguments, as is in particular the case for the standard and much used log separable utility function. This is because, as claimed above, non-neutrality does not depend on the shape of the utility function. The separability assumption, which allows for an explicit solution to the general equilibrium, is frequently encountered in the literature. In this case $(U_{cm} = 0)$, the real short rate becomes:

$$r = \mu_{\eta} - \left( -\frac{cU_{cc}}{U_c} \right) \sigma_{\eta} \sigma_c$$

(20)

Interestingly, $r$ has the same structure as the real short rate present in non-monetary economies such as CIR’s, with the essential difference that (endogenous) consumption here is affected by monetary shocks. An important consequence of (20) is that partial equilibrium monetary models with separable utility function and exogenous consumption and output must yield a real rate that is affected by real shocks only. This is for instance the case in Bakshi and Chen (1996). More generally, these partial equilibrium models cannot yield money non-neutrality endogenously.

As to the nominal interest rate and the inflation risk premium, they are the subjects of the following two propositions.

**Proposition 5 (nominal interest rate):**

*In equilibrium, the nominal short-term rate of interest is equal to:*

$$R = \frac{U_m}{U_c}$$

(21)

According to this well-known result, the nominal rate is equal to the marginal rate of substitution between real money balances and consumption. The cost of holding one additional unit of money is the opportunity cost $R$ (dollars), while the opportunity cost of consuming (destroying) one unit of consumption/investment good is one (dollar). The

---

19 See for instance Fama and Farber (1979), Poncet (1983), or Bakshi and Chen (1996).
cost ratio, \( R \), must at the optimum be equal to the ratio of marginal utilities\(^{20}\). It is in general affected by both monetary and real shocks.

**Proposition 6 (inflation risk premium):**

The inflation risk premium, defined as

\[
\varepsilon \equiv R + E_t \left[ \frac{dp^{-1}}{p^{-1}} \right] - r,
\]

is equal to:

\[
\varepsilon = -\frac{cU_{cc}}{U_c} \left( -\sigma_p \right)' \sigma_c - \frac{mU_{cm}}{U_c} \left( -\sigma_p \right)' \sigma_m
\]

(22)

The inflation risk premium is the reward to be granted to an investor, in addition to the real rate and to the expected rate of loss of the purchasing power of money, to induce her to invest into a nominal, rather than a real, money market account. The nominal money market account is a risky asset in real terms and thus commands such a premium at equilibrium. Two risk premiums, one related to consumption risk and the other to real balances risk, compensate the investor for the risk borne on the real return on the nominal money market account \((-\sigma_p)\). Substituting for the volatility of the inflation rate given by equation (14), equation (22) clearly shows that, even when consumption and money are separable in the utility function \((U_{cm} = 0)\), monetary policy, and in particular monetary uncertainty, still plays a major role in fixing the equilibrium value of the inflation risk premium. We will give explicit expressions of this variable in Section 4 below.

### 4.3. Asset Pricing

This sub-section is devoted to the implications for asset pricing of our preceding analysis. Financial theory uses three alternative approaches to inter-temporal asset

\(^{20}\) In discrete time, the cost ratio is equal to \( R/(1+R) \). The denominator simplifies to one in continuous time.
pricing that will be examined in the following (non chronological) order: Breeden (1979)’s consumption-based capital asset pricing model (CCAPM), the pricing kernel (or stochastic discount factor) approach, and Merton (1973)’s intertemporal capital asset pricing model (ICAPM).

We have already shown how monetary shocks impact the real rate of interest and the inflation risk premium. The CCAPM for financial assets is given in the following

**Proposition 7 (CCAPM):**

In equilibrium, the expected real excess returns on contingent claims are equal to:

\[
\mu_s - 1_{H_r} = -\frac{cU cc}{U_c} \Sigma_s \sigma_c - \frac{mU cm}{U_c} \Sigma_s \sigma_m
\]

(23)

or, equivalently, to:

\[
\mu_s - 1_{H_r} = \left( -\frac{cU cc}{U_c} \frac{w c}{c} + \left( -\frac{mU cm}{U_c} \right) \Sigma_s \sigma_m \right)
\]

(24)

\[
-\left( -\frac{cU cc}{U_c} \frac{w c}{c} + \left( -\frac{mU cm}{U_c} \right) \frac{1 - \alpha}{\alpha} \Sigma_s \sigma_m \right)
\]

\[
-\left( -\frac{cU cc}{U_c} \frac{w c}{c} + \left( -\frac{mU cm}{U_c} \right) \frac{1}{1 - \alpha} \right) \Sigma_s \sigma_M
\]

\[
+ \left( -\frac{cU cc}{U_c} \frac{1}{c} \Sigma_s \Sigma_Y c_Y \right)
\]

The CCAPM (23) includes, in addition to the consumption related risk premium, a risk premium associated with real balances. When the premiums are made more explicit, as in equation (24), the impact of nominal monetary shocks becomes obvious. As in equation (19), the influence of inflation risk on the equilibrium risk premiums is
emphasized. The cross partial derivative $U_{cm}$ being negative, the well-known empirical property that assets whose returns bear a positive correlation with the inflation rate require a smaller risk premium is recovered.

An interesting implication of the preceding result follows. Assuming a separable utility function and an exogenous consumption process will again produce money neutrality. For instance, equation (23) does resemble Bakshi and Chen (1996)’s corresponding equation\textsuperscript{21}, with the notable exception however that our first covariance involves consumption instead of output. This in fact constitutes a major difference because output is exogenous to their model while consumption (and output) are endogenous to ours. To fully assess the importance of this difference, let us momentarily assume that money and consumption are separable in the individual’s utility function ($U_{cm} = 0$). In Bakshi and Chen (1996)’s model, money is superneutral since $c = y$ and their term $\left[ \frac{cU_{cc}}{U_c} \Sigma_s \sigma_y \right]$ thus is exogenous\textsuperscript{22}, in contrast with our term $\left[ \frac{cU_{cc}}{U_c} \Sigma_s \sigma_c \right]$, which is endogenous and depends on monetary policy.

According to the pricing kernel approach to intertemporal asset pricing, if an asset yields a stochastic dividend stream $x(t)$ from date $t$ up to infinity, its price $q(t)$ at $t$ is such that:

$$q(t)\Gamma(t) = E_{t}\left[ \int_{t}^{\infty} \Gamma(s)x(s)ds \right]$$

(25)

where the pricing kernel is defined by $\Gamma(t) \equiv U_c(t,c(t),m(t))$, with $U_c(.)$ the representative investor’s marginal utility of consumption.

\textsuperscript{21} See their equation (12) on page 248.
\textsuperscript{22} We use our notation.
The pricing kernel being a function of \( c \) and \( m \), it will be affected by both real and monetary factors. Consequently, money cannot be neutral vis-à-vis real prices (or returns) of financial assets. In particular, both the systematic and the surprise components of monetary policy will have a bearing on their equilibrium real excess returns. For instance, applying Itô’s lemma to the volatility of the pricing kernel yields:

\[
\sigma_r = \frac{cU_{cc}(c(t),m(t))}{U_c(c(t),m(t))} \sigma_c + \frac{mU_{cm}(c(t),m(t))}{U_c(c(t),m(t))} \sigma_m
\]  

(26)

where \( \sigma_c \) is given by equation (15) and \( \sigma_m \) is equal to \( (\sigma_M - \sigma_p) \) by Ito's lemma. It is worth noting that this conclusion still holds when money and consumption are separable in the individual’s utility function.

Lastly, according to Merton (1973)’s ICAPM, or multi-beta CAPM, equilibrium excess returns contain a risk premium for each and every state variable, in addition to the premium related to the market portfolio. These premiums, when properly accounted for in empirical research, make asset returns somewhat predictable. A huge amount of work has been devoted recently to assess the impact of asset return predictability on such issues as optimal portfolio choice and investors’ welfare. Not surprisingly, acting myopically or ignoring some state variables in optimal dynamic portfolio strategies is costly for an investor in terms of utility. In our setting, the ICAPM is generalized according to the following

**Proposition 8 (ICAPM):**

The expected real excess return on a particular financial asset \( i \) is equal to:

\[
\mu_{s_i} - r = \frac{S_{iw}}{S_i} w \sigma_w \lambda + \sum_{j=1}^{K} \frac{S_{iy_j}}{S_i} \sigma_{y_j} \lambda
\]  

(27)

where:

\]
\[ \lambda = \sigma_w - \frac{w J_{ww}}{J_w} + \sum_{j=1}^{K} \sigma_{Y_j} \frac{-J_{wY_j}}{J_w} \]  

This result shows that the return on each asset will include a risk premium associated with aggregate wealth (the market portfolio) and a risk premium for each and every state variable. Therefore it exhibits the standard structure, except that here each premium is (diversely) affected by both real and monetary parameters.

5. A specialized economy

In order to derive explicit solutions, in particular for the dynamics of real and nominal interest rates, and provide some additional insights as to the scope of our results, we now specialize the general economy presented in Sections 3 and 4. The simplifying assumptions are discussed first, and then the equilibrium is derived along with the values of real wealth, investment and consumption, the price level dynamics, the real and nominal interest rates and the inflation risk premium. The final subsection is devoted to the thorough analysis of the real and nominal interest rate dynamics in various situations.

5.1 The economy

We assume there are but two state variables \((K = 2)\), a technological one that affects, as in CIR, the production process only (called hereafter the “real” state variable), and a monetary one that influences the dynamics of the money supply exclusively (dubbed hereafter the “nominal” state variable). Assuming more than one state variable for each process would contribute nothing to the economic intuition underlying the results. Assuming that the technology is affected by monetary factors as well as real ones would not seriously impair the model’s tractability, but would blur the message regarding money non-neutrality. In fact, by assuming that the technology is not affected by money,
we make the presence of monetary factors in the dynamics of real variables truly endogenous. Nevertheless, our analysis does not preclude that the two state variables are correlated.

The exogenous technology and money supply processes obey respectively:

\[
\frac{d\eta(t)}{\eta(t)} = \mu_\eta Y_\eta(t)^{\lambda_\eta} dt + Y_\eta(t)^{\lambda_\eta} \sigma_\eta \, dZ(t) \tag{29}
\]

and

\[
\frac{dM(t)}{M(t)} = \mu_M Y_M(t)^{\lambda_M} dt + Y_M(t)^{\lambda_M} \sigma_M \, dZ(t) \tag{30}
\]

where \( \mu_\eta \) and \( \mu_M \) are positive constants, \( \sigma_\eta \) and \( \sigma_M \) are \((N + 2) \times 1\) vectors of positive constants, \( Z(t) \) is an \((N + 2) \times 1\) dimensional Wiener process in \( \mathbb{R}^{N+2} \), the constants \( \lambda_i \), for \( i = 1, \ldots, 4 \), are positive, and \( Y_\eta \) and \( Y_M \), respectively, are the real and the nominal state variables assumed to follow:

\[
dY_\eta(t) = \mu_{Y_\eta} Y_\eta(t) dt + \sqrt{Y_\eta(t)} \sigma_{Y_\eta} \, dZ(t) \tag{31}
\]

and

\[
dY_M(t) = \mu_{Y_M} Y_M(t) dt + \sqrt{Y_M(t)} \sigma_{Y_M} \, dZ(t) \tag{32}
\]

where \( \mu_{Y_\eta} \) and \( \mu_{Y_M} \) are positive constants, \( \sigma_{Y_\eta} \) and \( \sigma_{Y_M} \) are \((N + 2) \times 1\) vectors of positive constants. We normalize \( \eta(0) = 1 \) and assume that \( M(0) \) is compatible with \( p(0) \).

We further assume that the representative investor has a log separable utility function, since it is a benchmark in finance theory:

\[
U(t, c(t), m(t)) = e^{-\rho t} \left[ \phi \ln c(t) + (1 - \phi) \ln m(t) \right] \tag{33}
\]
We can now derive the equilibrium values of all the relevant nominal and real variables.

5.2 The equilibrium

Solving for the model under the previous assumptions yields the following set of propositions.

**Proposition 9 (real investment):**

*In the specialized economy, the proportion of wealth invested in the technology is given by:*

\[ \alpha = f(Y_M) \]  \hspace{1cm} (34)

where \( f \) solves:

\[
\begin{align*}
  f' + \left( \frac{Y_M^{\lambda_4} \sqrt{Y_M \sigma_{Y_m}} \sigma_M - \mu_{Y_M} Y_M}{\rho + \mu_M Y_M^{2\lambda_4} \sigma_M^{\prime} \sigma_M} \right) f' \\
  - \frac{1}{2} \left( \frac{Y_M \sigma_{Y_m} \sigma_{Y_m} - \mu_{Y_M} Y_M^{2\lambda_4} \sigma_M^{\prime} \sigma_M}{\rho + \mu_M Y_M^{2\lambda_4} \sigma_M^{\prime} \sigma_M} \right) f''
\end{align*}
\]  \hspace{1cm} (35)

and where \( f' \) and \( f'' \) denote the first and second derivatives of \( f \), respectively.

The striking result is that the proportion of real wealth invested in equities depends exclusively on monetary factors (in addition, of course, to the utility parameters \( \phi \) and \( \rho \)), not on technological ones. This follows from assuming a log separable utility function.

A particular solution to (35) that is worth stressing occurs when the only state variable is the one that affects the technological process (\( K = 1 \)). The monetary process (30) then
reduces to a mere geometric Brownian motion and the investment to wealth proportion $\alpha$ simplifies to:

$$\alpha = \frac{\phi \rho + \mu_M - \sigma_M^2}{\rho + \mu_M - \sigma_M^2}$$

(36)

$\alpha$ then is a constant for given monetary parameters, but nonetheless remains affected by both the systematic and the surprise elements of monetary policy. In this subsection, we will refer to this important situation as “the special case”.

Equation (34) implies that a necessary condition for the investment to wealth proportion $\alpha$ to be stochastic is that the parameters of the money supply process are time varying through their dependence upon a state variable. Casual observation of the dynamics of the money supply and of the investment to wealth ratio over time in major economies shows that this is the rule rather than the exception. Therefore allowing for time variation in the drift and volatility of the money supply process, and thus in the (endogenous) dynamics of $\alpha$, is a desirable property. This impinges on the money neutrality and superneutrality issues. When the parameters of the money supply process and those of the change rate process for the ratio $\alpha$ are stochastic, monetary factors affect both the level and the change rate of real investment. When the parameters of the money supply process and the ratio $\alpha$ are constant, monetary policy still influences the optimal allocation of wealth between real money holdings and investment, as shown by equation (36). However, only the level of investment is affected, not its expected rate of growth. In any case, money can never be neutral vis-à-vis investment. Similar conclusions as to consumption and real wealth can be derived from the following proposition.
Proposition 10 (real wealth and consumption):

In the specialized economy, the representative agent’s real wealth is given by:

$$w(t) = w(0)e^{-\alpha t}e^{-\rho t}e^{-\xi_M(t)^{-1} \xi_\alpha(t)^{-1}}$$  \(37\)

where the processes \(\xi_M\) and \(\xi_\alpha\) are defined by:\(23\):

$$\frac{d\xi_M}{\xi_M} = \left(1 - \frac{f(Y_M)}{f'(Y_M)}\right) \sigma_M dt + \frac{f'(Y_M)}{f(Y_M)} \sqrt{\frac{\sigma_{\xi_M}}{\sigma_Y}} dZ$$  \(38\)

and

$$\frac{d\xi_\alpha}{\xi_\alpha} = -\frac{1 - f(Y_M)}{f'(Y_M)} \frac{f'(Y_M)}{f(Y_M)} Y_M^{-\lambda + 0.5} \sigma_{\xi_M} dt + \frac{f'(Y_M)}{f(Y_M)} \sqrt{\frac{\sigma_{\xi_M}}{\sigma_Y}} dZ$$  \(39\)

Furthermore, optimal consumption is strictly proportional to real wealth:

$$c = \phi w$$  \(40\)

As expected from the analysis in the preceding section, all real variables (wealth, consumption and investment) are affected by monetary factors, as money non-neutrality is preserved in our specialized setting. What is perhaps more surprising is that, even for log separable utilities, this influence remains quite complex. This is mainly due to the way money affects the proportion of real wealth invested in equity.

Equilibrium real wealth is affected by monetary policy through the processes \(\xi_M(t)\) and \(\xi_\alpha(t)\) which are both affected by monetary factors only. \(\xi_M(t)\) is a martingale related to money supply uncertainty while \(\xi_\alpha(t)\) is related to the uncertainty regarding the investment to wealth ratio due to the nominal state variable affecting the parameters of

\(23\) \(\xi_M(0)\) is equal to one.
the money supply process. Thus, monetary uncertainty influences equilibrium in the real sector through two distinct channels. Note however that only one of them is sufficient to insure that money is not superneutral vis-à-vis wealth and consumption. Indeed, if $\sigma_{YM} = 0$, the process $\xi_\alpha(t)$ reduces to the constant one but the process $\xi_M(t)$ still is a martingale, and, if $\sigma_M = 0$, the process $\xi_M(t)$ reduces to the constant one, but $\xi_\alpha(t)$ is a martingale. Only if the money growth rate and the nominal state variable are both deterministic ($\sigma_M = \sigma_{YM} = 0$) is money superneutral vis-à-vis real wealth and, because of equation (40), consumption (but not investment).

This money non-neutrality property of our equilibrium stems from the inflation rate being stochastic. The latter randomness has in fact three sources: money supply volatility, technological uncertainty and the volatility of the investment to wealth ratio $\alpha$ due to the influence of the nominal state variable $Y_M$. Even though the money supply and the technological return processes are deterministic, the price level remains stochastic. This is clear by inspection of equation (35) with $\sigma_M = 0$, where $f'$ and $f''$ do not vanish so that $\alpha$ is not a constant, and the real demand for money evolves over time in a random manner. Since $\alpha$ depends on the drift of the money supply, money is not neutral.

Similar results hold for the behavior of the price level to which we turn now.

---

24 See equation (41) below. Note that real wealth given by equation (37) is affected by the same sources of randomness.
**Proposition 11 (price level):**

*In the specialized economy, the equilibrium price level is given by:*

\[
p(t) = e^{\alpha t} \frac{M(t)}{(1-f(Y)(t))w(0)\eta(t)} \xi_M(t) \xi_a(t)
\]  

(41)

The price level is by definition the ratio of nominal money stock \(M(t)\) over real money balances \([e^{-\rho t}(1-f(Y))w(0)\eta(t)\xi_M(t)^{-1}\xi_a(t)^{-1}]\), the complement to wealth of real investment. Since monetary policy affects both the latter and \(M(t)\), the relationship between the price level dynamics and the money supply process is highly non-linear.

The one-to-one relationship generally assumed in the literature between the inflation rate and the expected money growth rate thus does not hold in a general equilibrium model, even in our specialized version of the economy. To illustrate, in the special case where equation (36) holds, one can derive explicitly the expectation of the inflation rate and its volatility, which read respectively:

\[
\mu_p = \rho - \left(-\mu_M + \sigma_M \sigma_M\right) + \left(-\mu_\eta + \sigma_\eta \sigma_\eta\right) + \frac{\rho + \mu_M - \sigma_M \sigma_M}{\phi + \mu_M - \sigma_M \sigma_M} \left(\sigma_M \sigma_M - \sigma_\eta \sigma_\eta\right)
\]

(42)

\[
\sigma_p = \frac{\rho + \mu_M - \sigma_M \sigma_M}{\phi + \mu_M - \sigma_M \sigma_M} \sigma_M - \sigma_\eta
\]

(43)

It is clear that both parameters depend on the two components of monetary policy. These findings, whose in-depth analysis is beyond the scope of this paper, have obvious implications for the design and conduct of monetary policy.
Proposition 12 (nominal interest rate):

_in the specialized economy, the equilibrium nominal short-term rate is equal to:

\[ R = \frac{(1-\phi)\rho}{1-f(Y_M)} \]  

(44)

_in the special case of a constant alpha (for a given set of monetary parameters), its expression simplifies further to:

\[ R = \rho + \mu_M - \sigma_M' \sigma_M \]  

(45)

The nominal rate thus is independent of technological parameters and is affected by monetary factors only (in addition to the investor’s preference parameters). In the special case of a constant alpha, R is equal to the representative individual’s impatience rate minus the expected relative change in 1/M (equal to \(-\mu_M + \sigma_M' \sigma_M \)). As expected, the nominal rate increases with an expansionary monetary policy, i.e. a decrease in the anticipated growth rate of 1/M\(^2\). This is consistent with the empirical evidence recently documented by Evans and Marshall (2001) according to which most of the long-term variability in nominal interest rates of all maturities is due to macroeconomic impulses. More specifically, monetary policy shocks impact the slope of the term structure in a consistent and significant manner, while fiscal policy impulses do not seem to have any significant influence.

---

\(^{25}\) The model thus does not allow for a (Keynesian) liquidity effect that temporarily decreases the nominal rate before inflationary expectations raise it above its initial level. Some modern neo-Keynesian models, reviewed in Gali (2000), also share this property. Empirically, numerous recent works failed to uncover a liquidity effect. The nominal interest rate is found either to be almost invariant to changes in a given monetary aggregate or to vary directly with the latter. By contrast, Bernanke and Mihov (1998) argued in favor of a liquidity effect. However, their findings have not been confirmed and may have been biased by the methodology used to measure monetary shocks.
**Proposition 13 (real interest rate):**

**In the specialized economy, the equilibrium real short-term rate is equal to:**

\[
r = \mu_{\eta} - Y_{\eta}^{\lambda_1} - Y_{\eta}^{2\lambda_2} \sigma_{\eta} \sigma_{\eta} + \frac{1 - f(YM)}{f(YM)} Y_{\eta}^{\lambda_1} Y_{\eta}^{\lambda_2} \sigma_{\eta} \sigma_{\eta} \sigma_{YM} + \frac{f'(YM)}{f(YM)} \sqrt{Y_{\eta}^{\lambda_2} \sigma_{\eta} \sigma_{YM}} \] (46)

**In the special case of a constant alpha, its expression simplifies to:**

\[
r = \mu_{\eta} - \sigma_{\eta} \sigma_{\eta} + \frac{(1 - \phi)\rho}{\phi \rho + \mu_M - \sigma_M \sigma_M} \sigma_{\eta} \sigma_{\eta} \] (47)

The real rate depends, as expected, on two “real” factors, namely the representative investor’s preference parameters, \(\rho\) and \(\phi\), and the characteristics of the technological returns, \(\mu_\eta\) and \(\sigma_\eta\). It also depends on monetary and “nominal” factors, namely the parameters of the money growth rate and those of the nominal state variable. Money non-neutrality is brought about by two sources of risk, the covariance between technological returns and the money growth rate on the one hand, and the covariance between technological returns and changes in the nominal state variable (that also influence the monetary policy parameters) on the other. It may be noted that only one source of risk is in fact necessary to achieve non-neutrality. In any case, money affects the real interest rate through the interaction between monetary (or nominal) and real uncertainties. The interpretation of this result is straightforward. The real rate is equal to the expected return from the technology minus the required risk premium, the latter depending on the covariance between consumption and equity returns. For money not to be neutral, consumption must be affected by monetary factors so that at least one source of monetary risk exists.

The sign and magnitude of the impact of monetary impulses on the real short rate level are directly related to the sign and magnitude of the covariances above. For instance, in the special case of equation (47), the change in \(r\) due to an increase in the average
monetary growth rate $\mu_M$ has a sign opposite to the sign of the covariance $\sigma_\eta'\sigma_M$ for the following reason\textsuperscript{26}. Assuming for example that the return on real investment is positively correlated with the money growth rate, an increase in the latter will make the equilibrium rate $r$ decrease since the demand for real savings will increase due to a positive wealth effect.

This feature is vindicated by the result that, ceteris paribus, the real rate covaries negatively with the expected inflation rate. This can be seen by eliminating the term $(\mu_\eta - \sigma_\eta'\sigma_\eta)$ from equation (47) using equation (42), which yields the term $-\mu_p$ on the RHS of equation (47). This finding is consistent with an empirical study by Pennachi (1991) who showed, using NBER-ASA survey inflation forecasts and U.S. Treasury-bill prices, that the real interest rate and the inflation rate follow joint diffusion processes and that the instantaneous real rate and the instantaneous rate of expected inflation are significantly negatively correlated.

In the special case where the money supply is deterministic, equation (47) simplifies to:

$$r = \mu_\eta - \sigma_\eta'\sigma_\eta.$$  

We recover money superneutrality with respect to $r$. Since real wealth and consumption are independent of the money supply, the risk premium associated with the covariance between consumption and real investment returns does not depend on the money growth rate. Another important implication of (47) is that a sufficient condition for the real rate not to be affected by monetary policy is that real and monetary shocks are uncorrelated. Recall that money affects the real rate through the risk premium required on the technology returns. When equity returns and money growth are not correlated, money leaves this risk premium, hence the real interest rate, unaffected.

\textsuperscript{26} The derivative $\frac{\partial r}{\partial \mu_M}$ is equal to $\frac{-(1 - \phi)\sigma_\eta'\sigma_M}{\left(\phi\rho + \mu_M - \sigma_M'\sigma_M\right)^2}$. 

41
**Proposition 14 (inflation risk premium):**

In the specialized economy, the equilibrium inflation risk premium is given by:

\[
\varepsilon = -\frac{2 - f(Y_M)}{f(Y_M)} Y_\eta(t)^{\lambda_2} Y_M(t)^{\lambda_4} \sigma_M \sigma_\eta + Y_\eta^{2\lambda_2} \sigma_\eta \sigma_\eta + \frac{1 - f(Y_M)}{f(Y_M)^2} Y_M^{2\lambda_4} \sigma_M \sigma_M
\]

\[
-\frac{2 - f(Y_M)}{1 - f(Y_M)} Y_\eta \frac{f'(Y_M)}{f(Y_M)} \sqrt{Y_M} \sigma_M \sigma_\eta + \frac{2}{f(Y_M)} Y_M(t)^{\lambda_4} \frac{f'(Y_M)}{f(Y_M)} \sqrt{Y_M} \sigma_M \sigma_\eta
\]

\[
+ \frac{1}{1 - f(Y_M)} \left( \frac{f'(Y_M)}{f(Y_M)} \right)^2 Y_M \sigma_M \sigma_M
\]

In the special case of a constant \(\alpha\), it reduces to:

\[
\varepsilon = -\frac{(2 - \phi)\rho + \mu_M - \sigma_M'}{\phi \rho + \mu_M - \sigma_M} \sigma_M \sigma_\eta + \sigma_\eta \sigma_\eta
\]

\[
+ \frac{(1 - \phi)\rho + \mu_M - \sigma_M \sigma_M'}{\phi \rho + \mu_M - \sigma_M \sigma_M} \sigma_M \sigma_M
\]

As for the real rate, money non-neutrality is brought about by two sources of risk, the stochastic money growth rate on the one hand, and stochastic changes in the nominal state variable (that also influence the monetary policy parameters) on the other. Again, only one source of risk is in fact necessary to achieve non-neutrality. However, while for money to affect the real interest rate there should be an interaction between monetary and real uncertainties (non-zero covariances), this is not necessary for the inflation risk premium. The variance of the money supply process and/or the variance of the state variable affect directly this premium.

The inflation risk premium depends on real factors through the variance of the technology productivity. Since an increase in the latter enlarges the variance of the inflation rate, \(\varepsilon\) is a positive function of \(\sigma_\eta \sigma_\eta\). As to monetary policy, it affects the premium through three channels: the systematic part \(\mu_M\) that bears on \(\alpha\), and the surprise
parts $\sigma_M$ and $\sigma_\eta \sigma_M$. Although equation (49) is complex, the inflation risk premium is expected to be positive and is obviously so when the covariance between equity returns and money growth ($\sigma_\eta \sigma_M$) is non-positive. Even if monetary policy were deterministic, the premium would still be positive under output uncertainty, because, as observed previously, the inflation rate would remain stochastic. In addition, were real returns not stochastic ($\sigma_\eta = 0$) or not correlated with monetary growth ($\sigma_\eta \sigma_M = 0$), the inflation risk premium would remain nonetheless affected by the variance of the money growth rate, as intuition suggests.

5.3 Interest rate dynamics

We now turn to the in-depth analysis of interest rate dynamics. Here, we examine a case that is more general than that of the preceding subsection where $\alpha$ was a constant for a given set of monetary parameters. This case now allows for both the real and the nominal state variables to be present ($K = 2$) but is restricted to the situation where equation (35) can be meaningfully approximated by:

$$\approx = \alpha \lambda \lambda$$

One way to assess that this approximation is reasonable is to check that, in absence of state variables, equation (50) does reduce to the exact solution (36).

Equation (44) for the nominal rate then reads:

$$R = \rho \mu_M Y_M^{\lambda_3} - Y_M^{2k_4} \sigma_M \sigma_M$$

which is a slight but important generalization of equation (45). Interestingly, the level of the nominal rate is independent of the parameters of the state variable processes, as is
the level of the real rate in CIR. However, it does logically depend on the level of the nominal state variable.

The literature regarding the proper way to model the dynamics of nominal short term rates is vast and expanding rapidly. This is no wonder as interest rates lie at the core of many economic and financial issues, the solution to which require a reliable model for the stochastic behavior of interest rates. However, as stated in the introduction, no consensus has been reached yet. In particular, the standard and popular affine term structure models such as Vasicek (1977), CIR (1985b) or Duffie and Kan (1996), which are still much in use among practitioners, have been shown to perform rather poorly by Chan et al. (1992), possibly because of the implicit restrictions they impose on the term structure volatility. The new strands of models developed by Aït-Sahalia (1996), Stanton (1997), Conley et al. (1997), and Boudoukh et al. (1999) among others that apply non-parametric estimation techniques to nominal short term rates reveal strong non-linearities in the drift functions of those rates. The various quadratic term structure models recently offered by Ahn et al. (2002) and Leippold and Wu (2000) account for these non-linearities as well as non-linearities in the diffusion functions. At the empirical level, two approaches have been adopted [see Smith (2000) for a comparison and a review]: regime switching for the underlying state variable(s) or stochastic volatility. According to Ahn and Gao (1999), Smith (2000), Brandt and Yaron (2001), and Ahn et al. (2002), whose model encompasses older models by Longstaff (1989), Longstaff and Schwartz (1992), Beaglehole and Tenney (1992) and Constantinides (1992), quadratic or more generally non-linear models seem to outperform affine models, although Chapman and Pearson (2000) tend to disagree and Dai and Singleton (2001) offer a more mitigated evidence.

In what follows, we show that our model is general enough to embed all the dynamics for the nominal short-term rate of interest proposed above. These dynamics depend in an essential way on the values taken by the $\lambda_3$ and $\lambda_4$ parameters. We provide three types of models of various generality. We start from a model that remains in the spirit of CIR,
and then generalize it to the case where the parameters of the money supply process are proportional to a function of the state variable rather than to the state variable itself as in CIR. Finally, we relax the CIR’s assumption of a square root process for the nominal state variable.

5.3.1. CIR-style dynamics

Setting \( \lambda_3 = 1 \) and \( \lambda_4 = 0.5 \) in a CIR-like manner leads to the following dynamics for the nominal short-term rate:

**Proposition 15 (affine structure for the nominal rate):**

Assuming \( \lambda_3 = 1 \) and \( \lambda_4 = 0.5 \), the nominal short-term rate is equal to:

\[
R(t) = \rho + \left( \mu_M - \sigma_M \nu_M \right) Y_M(t) \tag{52}
\]

and its dynamics obeys:

\[
dR(t) = \kappa_1 (R(t) - \rho) dt + \sqrt{\kappa_2 (R(t) - \rho)} dz(t) \tag{53}
\]

where \( z(t) \) is a standard one-dimensional Wiener process and:

\[
\kappa_1 \equiv \mu_{Y_M} \\
\kappa_2 \equiv \sigma_{Y_M} \left( \mu_M - \sigma_M \nu_M \right)
\]

It follows that when the monetary supply process follows a dynamics such that both the drift and the variance are proportional to the nominal state variable, the nominal short rate essentially inherits the dynamics of the latter. The square root process (53) resembles that of CIR, but is slightly more general due to the presence in \( \kappa_2 \) of the parameters of the money supply process. This provides a long awaited theoretical justification to the dozens of papers adopting standard affine models that assume that the
dynamics of the nominal short-term rate is given by CIR’s dynamics, although CIR derived their results for the real rate. In our setting, the latter rate and the inflation risk premium can be explicitly computed, as shown in the following proposition.

**Proposition 16 (real rate and inflation risk premium):**

Assuming \( \lambda_1 = \lambda_3 = 1 \) and \( \lambda_2 = \lambda_4 = 0.5 \), the real short-term rate is equal to:

\[
 r = \left( \mu_{\eta} - \sigma_{\eta} \sigma_{\eta} \right) Y_{\eta}
 + \frac{(1 - \phi)p}{\phi + (\mu_M - \sigma_M \sigma_M)Y_M} \left( \sigma_{M} \sigma_{M} + \left( \mu_M - \sigma_M \sigma_M \right) \sigma_{\eta} \sigma_{Y_M} \right) \sqrt{Y_M Y_{\eta}}
\]

and its dynamics obeys:

\[
 dr = h(r, R) dt + \psi(r, R) d\bar{Z}
\]

where \( \bar{Z} \) is a standard one-dimensional Wiener process and:

\[
 \psi(r, R) = \left( \mu_{\eta} - \sigma_{\eta} \sigma_{\eta} \right) \sqrt{a(r, R)} + \frac{1}{2} \frac{g(R)}{\mu_M - \sigma_M \sigma_M} \left( R - \rho \right) \sigma_{Y_M} \sigma_{Y_M}
 + 2 \left( \mu_{\eta} - \sigma_{\eta} \sigma_{\eta} \right) \sqrt{a(r, R)} + \frac{1}{2} \frac{g(R)}{\mu_M - \sigma_M \sigma_M} \sigma_{Y_M} \sigma_{Y_M}

h(r, R) = \left( \mu_{\eta} - \sigma_{\eta} \sigma_{\eta} \right) a(r, R) + \frac{1}{2} \frac{g(R)}{\mu_M - \sigma_M \sigma_M} \left( R - \rho \right) \sigma_{Y_M} \sigma_{Y_M}
 + \frac{1}{2} \sqrt{a(r, R)} g_{\eta \eta} + \frac{1}{2} \frac{g''(R)}{\mu_M - \sigma_M \sigma_M} \left( R - \rho \right) \sigma_{Y_M} \sigma_{Y_M}
 - \frac{1}{8} g'(\sqrt{a(r, R)})^{-1} \sigma_{Y_M} \sigma_{Y_M} + \frac{1}{2} \sqrt{a(r, R)} g_{\eta \eta} \sigma_{Y_M} \sigma_{Y_M}

\[
g(R) = \frac{(1 - \phi)p}{R - (1 - \phi)p} \left( \sigma_{\eta} \sigma_{\eta} + \left( \mu_M - \sigma_M \sigma_M \right) \sigma_{\eta} \sigma_{Y_M} \right) \sqrt{R - \rho}
\]

\( \mu, \sigma, \rho, \phi, \eta, \sigma, Y, M, R, 1, 2, 3, 4 \)
and \( a(r,R) \) is the positive root of:

\[
0 = -r + \left( \mu - \sigma \right) x + g(R)\sqrt{x}
\]

Under the same set of assumptions, the inflation risk premium is equal to\(^{27}\):

\[
\varepsilon = -\frac{(1-\phi)\rho + R}{(\phi-1)\rho + R} \sqrt{a(r,R)} \sqrt{\frac{R-\rho}{\mu - \sigma}} \sigma_\eta \sigma_\eta + a(r,R)\sigma_\eta \sigma_\eta
\]

\[
+ \frac{(1-\phi)(R-\rho)}{(\phi-1)\rho + R} \sigma_M \sigma_M + 2 \frac{(1-\phi)\rho(R-\rho)}{(\phi-1)\rho + R} \sigma_{\gamma M} \sigma_M
\]

\[
- \frac{(\mu_M - \sigma_M \sigma_M)}{R((\phi-1)\rho + R)} \sqrt{a(r,R)} \sqrt{\frac{R-\rho}{\mu - \sigma}} \sigma_\gamma \eta
\]

\[
+ \frac{(1-\phi)\rho(\mu_M - \sigma_M \sigma_M)}{R((\phi-1)\rho + R)} \sigma_{\gamma M} \sigma_M
\]

Interestingly, the real rate process does not recover the linearity found for the nominal rate process. Hence, somewhat paradoxically, CIR’s square root process is more appropriate for modeling the nominal rate than for modeling the real one, in spite of their economy being purely real.

5.3.2. Generalized CIR dynamics

We now assume more generally that the drift and the variance of the money supply process are proportional to a function of the state variable rather than to the state variable itself. As will be shown, the interest rate dynamics changes noticeably. We focus here on the nominal rate, since it is the concern of empirical research and professional applications and the expressions for the real rate become rapidly too complex. We provide two different examples leading to a non-linear and a quadratic structure for the nominal rate, respectively.

\(^{27}\) Its dynamics can also be computed but the result is intractable.
Proposition 17 (non-linear structure):

Assuming $\lambda_3 = 2\lambda_4$, the nominal short-term rate becomes:

$$ R(t) = \rho + \left( \mu_M - \sigma_M' \right) \gamma_M(t)^{\gamma_3} $$

(57)

and its dynamics obeys:

$$ dR(t) = \left( \kappa_5 + \kappa_6 R(t) + \kappa_7 (R(t) - \rho)^{\gamma_5} \right) dt + \sqrt{\kappa_8 (R(t) - \rho)^{\gamma_6}} \, d\tilde{z}(t) $$

(58)

where $\tilde{z}$ is a standard one-dimensional Wiener process and:

$$ \gamma \equiv 1 - \frac{1}{\lambda_3} $$

$$ \kappa_5 \equiv -\lambda_3 \mu_{YM} \rho $$

$$ \kappa_6 \equiv \lambda_3 \mu_{YM} $$

$$ \kappa_7 \equiv \frac{1}{2} \lambda_3 \left( \lambda_3 - 1 \right) \sigma_{YM} \gamma_M' \left( \mu_M - \sigma_M' \right)^{\frac{1}{\gamma_3}} $$

$$ \kappa_8 \equiv \lambda_3 \sigma_{YM} \gamma_M' \left( \mu_M - \sigma_M' \right)^{\frac{1}{\gamma_3}} $$

The dynamics of the nominal interest rate is far more complicated than the simple square root process. In particular, the drift and the volatility are highly non-linear functions of the instantaneous rate. Therefore, without assuming exogenous non-linearities in the spot rate or in the state variables, our model delivers non-linearities in the nominal rate dynamics in a simple way. An interesting result is that the drift and the volatility of the nominal rate process react differently to its current value. Several recent non-linear models such as Ahn and Gao (1999) or Ang and Beckaert (2002) thus are recovered.

Our model also embeds the important quadratic term structure models recently developed by Leippold and Wu (2000) and Ahn, Dittmar and Gallant (2002) and
empirically tested by Brandt and Yaron (2001). Ahn, Dittmar and Gallant (2002) provided a theoretical foundation for such models based on equilibrium properties of a real economy. The following proposition shows that this is but a particular case of our monetary economy.

**Proposition 18 (quadratic structure):**

*Assuming $\lambda_1 = \lambda_4 = 1$, the nominal short-term rate is equal to:*

\[
R(t) = \rho + \mu_M Y_M(t) - \sigma_M' \sigma_M Y_M(t)^2
\]  \hspace{1cm} (59)

*and, if the parameters of the money supply process are such that there is only one positive root to (59), its dynamics is such that:

\[
dR(t) = \left(\kappa_9 + \kappa_{10} R(t) + \kappa_{11} \sqrt{\kappa_{12} + \kappa_{13} R(t)}\right)dt \\
+ \left[\kappa_{14} + 2R(t) + \kappa_{15} \sqrt{\kappa_{12} + \kappa_{13} R(t)}\left[\kappa_{16} \sqrt{\kappa_{12} + \kappa_{13} R(t)} - \kappa_{15}\right]\right]^{\frac{1}{2}} \kappa_{17} d\hat{z}(t)
\]  \hspace{1cm} (60)

*where $\hat{z}$ is a standard one-dimensional Wiener process and:*
\[ \kappa_9 \equiv -2 \rho \mu'_{Y_M} - \left( \sigma'_M \sigma_M \sigma'_{Y_M} \sigma_{Y_M} + \mu_M \mu'_{Y_M} \right) \frac{\mu_M}{2 \sigma_M \sigma_M} \]
\[ \kappa_{10} \equiv 2 \mu'_{Y_M} \]
\[ \kappa_{11} \equiv - \left( \sigma'_M \sigma_M \sigma'_{Y_M} \sigma_{Y_M} + \mu_M \mu'_{Y_M} \right) \frac{1}{2 \sigma_M \sigma_M} \]
\[ \kappa_{12} \equiv \mu_M^2 + 4 \sigma_M \sigma'_{Y_M} \sigma'_{M} \rho \]
\[ \kappa_{13} \equiv -4 \sigma_M \sigma_M \rho \]
\[ \kappa_{14} \equiv -2 \rho - \frac{\mu_M^2}{2 \sigma_M \sigma_M} \]
\[ \kappa_{15} \equiv - \frac{\mu_M}{2 \sigma_M \sigma_M} \]
\[ \kappa_{16} \equiv \frac{1}{2 \sigma_M \sigma_M} \]
\[ \kappa_{17} \equiv \sqrt{\sigma'_{Y_M} \sigma_{Y_M}} \]

One advantage of the specification (60) is that it remains a one-factor model, although it exhibits a quadratic structure for the level of \( R \).

5.3.3. Other dynamics

In fact, our framework is rich enough to generate for the interest rate dynamics many classes of processes. For instance, an alternative to the results above can be obtained by relaxing CIR’s assumption of a square root process for the state variable. This assumption was required in CIR to ensure a positive interest rate: since, in equilibrium, their (real) interest rate is proportional to their (real) state variable, the latter must be positive almost surely. It turns out that other processes for the state variables not only lead to positive (nominal) interest rates but also to empirically interesting rate dynamics. For instance, let us assume that the monetary state variable follows:

\[ dY_M(t) = \mu_{Y_M} Y_M(t) dt + Y_M(t) \sigma_{Y_M} \sigma_{Y_M}^t dZ(t) \]  
(61)
while the general dynamics for the technology return and the money supply growth rate are preserved. Then the level of the nominal rate is still given by equation (57) but its dynamics is quite different. We provide two specific examples. The first one is in the spirit of the lognormal model of Miltersen, Sandman and Sonderman (1997) where simple (not compounded) forward rates follow lognormal diffusions.

**Proposition 19 (lognormal rate):**

Assuming \( \lambda_3 = 2 \lambda_4 \), the nominal rate is given by:

\[
R(t) = \rho + \left( \mu_M - \sigma_M \sigma_M \right) \gamma(t)
\]

and its dynamics is such that:

\[
dR(t) = \kappa_{18} (R(t) - \rho) dt + \kappa_{19} (R(t) - \rho) d\hat{z}(t)
\]

where \( \hat{z} \) is a standard one-dimensional Wiener process and:

\[
\kappa_{18} \equiv \lambda_3 \mu_M + \frac{1}{2} \lambda_3 (\lambda_3 - 1) \sigma_M \sigma_M \\
\kappa_{19} \equiv \lambda_3 \sqrt{\gamma_M \gamma_M}
\]

An important consequence is that the drift and the variance of the dynamic process are linear functions of the nominal rate level even though the rate itself is not a linear function of the state variable. This is not the case in equation (58), although equations (57) and (62) are identical.
Proposition 20 (linear-quadratic):

Assuming $\lambda_3 = \lambda_4$, the nominal interest rate is given by:

$$R(t) = \rho + \mu M Y_M(t)^{\lambda_3} - \sigma M \sigma M Y_M(t)^{2\lambda_3}$$  \hspace{1cm} (64)

and its dynamics is such that:

$$dR(t) = (\kappa_{20} + \kappa_{21}v(t) + \kappa_{22}R(t))dt + v(t)\sigma_{Y_{M}} dZ(t)$$  \hspace{1cm} (65)

where:

$$v(t) = \lambda_3\mu M Y_M(t)^{\lambda_3} - 2\lambda_3\sigma M \sigma M Y_M(t)^{2\lambda_3}$$

$$\kappa_{20} \equiv \frac{1}{2}\lambda_3^2\sigma_{Y_{M}}^2 \mu M$$

$$\kappa_{21} \equiv \mu M + \frac{1}{2}(\lambda_3 - 1) + \frac{1}{2}\lambda_3\sigma_{Y_{M}}^2$$

$$\kappa_{22} \equiv -\frac{1}{2}\lambda_3^2\sigma_{Y_{M}}^2$$

The dynamics (65) for the nominal rate is reminiscent of the result obtained by Longstaff and Schwartz (1992). The major difference is that their interest rate is the real one. Note that their model already embeds CIR’s model in that the term structure is driven by two factors (the rate level and its volatility) instead of one. In equation (65), our two factors are directly linked to the sources of uncertainty, namely the two monetary policy parameters $\mu M$ and $\sigma M$ affected by the nominal state variable $Y_M$. This is, again, consistent with Evans and Marshall’s (2001) findings that monetary impulses have a sizeable and lasting influence on the nominal term structure.
6. Conclusion

We have derived the general equilibrium dynamics of the main real and nominal aggregate variables in a monetary economy affected by technological and monetary shocks. The level and dynamics of any real variable, in particular the short-term real rate of interest, is inherently driven by both monetary and real factors. Money non-neutrality thus is generic, as it does not stem from any friction such as price or wage stickiness, asymmetric information and restricted participation or from a particular utility function. Non-neutrality obtains because the ex ante cost of real money holdings is random due to inflation uncertainty. In a specialized version of this economy in which the state variables follow particular processes, and the representative investor has a log separable utility function, we have explicitly derived the level and dynamics of the short-term real and nominal interest rates. These two kinds of rates in fact behave in very different manners, as the inflation risk premium is diversely affected by real and nominal shocks. The processes obtained for the nominal interest rate encompass most of the dynamics offered in the literature, from the standard affine models to the recent quadratic and non-linear models, and lead to new, more general, nominal term structures.

The proposed setting is sufficiently flexible so that many interesting issues of interest for both academics and practitioners can be addressed. At the microeconomic level, these range from fixed income instrument pricing, option pricing and hedging, asset-liability management, value-at-risk assessment and other interest risk measurements to the valuation of floaters, interest rate and currency swaps, forwards and futures, and swaptions, to name a few. At the macroeconomic level, they range from the conduct of monetary policy and its impact on real income, investment, consumption and wealth and on the inflation, interest and exchange rates to the relationship between asset returns and inflation, the equity premium puzzle, the current hotly debated stock return predictability issue, and, last but foremost, the term structure estimating issue.
This work could be extended in a number of ways. First, instead of adopting the money-in-the-utility approach, one could consider a cash-in-advance economy with a credit good in addition to the cash good. Second, an explicit reaction function on the part of the monetary authorities could be modeled rather than assuming a purely exogenous process for the money supply. This could allow for the comparative study of the influence of various policy rules on the aggregate real and nominal variables deemed relevant. Third, a government sector with autonomous expenditures and nominal taxes could be introduced, for instance to assess the impact of fiscal policies on the real and nominal yield curves.
**Mathematical appendix**

Proof of Propositions 1 and 2:

At equilibrium, given equation (9) and the three market clearing conditions (i)-(iii) given at the outset of Section 3, the optimal wealth dynamics writes:

\[
\frac{dw}{w} = \mu_w dt + \sigma_w \, dZ
\]  

(A1)

where

\[
\mu_w = \alpha \mu_\eta - \frac{c}{w} - (1 - \alpha)(\mu_p - \sigma_p \sigma'_p) \\
\sigma_w = \alpha \sigma_\eta - (1 - \alpha) \sigma_p
\]  

(A2)

At equilibrium, we have: \( m = (1 - \alpha)w \). The price level dynamics is obtained by applying Ito's lemma to:

\[ p = \frac{M}{m} = \frac{M}{(1 - \alpha)w} \]

This yields:

\[
\mu_p = \mu_M - \mu_w + \frac{\alpha}{1 - \alpha} \mu_\alpha + \frac{\alpha}{1 - \alpha} \sigma'_\alpha \\
+ \left(\sigma_w - \frac{\alpha}{1 - \alpha} \sigma_\alpha\right) \left(\sigma_w - \frac{\alpha}{1 - \alpha} \sigma_\alpha\right) - \sigma'_M \left(\sigma_w - \frac{\alpha}{1 - \alpha} \sigma_\alpha\right) \\
\sigma_p = \sigma_M - \sigma_w + \frac{\alpha}{1 - \alpha} \sigma_\alpha
\]  

(A3)

Solving for the system (A2) and (A3) yields equations (11), (12), (14) and (15).
Proof of Proposition 3:

Let \( J(t, w(t), Y(t)) \) be the representative agent’s value (indirect utility of wealth) function and let \( LJ \) be the differential generator of \( J \). We assume that \( J \) exists and is an increasing and strictly concave function of \( w \). Define \( \psi \equiv LJ + U \). Then deriving the Bellman - Hamilton - Jacobi equation:

\[
0 = \operatorname{Max} \left( U + J_w w \mu_w + J_{wY} w \mu_Y + \frac{1}{2} J_{ww} w^2 \sigma_w ' \sigma_w + \frac{1}{2} J_{wY} \Sigma_Y \Sigma_Y ' + J_{wY} \Sigma_Y \sigma_w \right)
\]

with respect to the control variables yields the following necessary and sufficient conditions for optimality (with the usual notations for the partial first and second derivatives of the value function):

\[
\psi_c = U_c - J_w \leq 0 \quad (A4)
\]

\[
c \psi_c = 0 \quad (A5)
\]

\[
\psi_m = U_m - \left( r + \mu_p - \sigma_p ' \sigma_p \right) J_w - \left( w \sigma_\eta ' \sigma_\eta + w \theta \Sigma_s \sigma_p - \left( \delta w + m \right) \sigma_p ' \sigma_p \right) J_{ww}
\]

\[
- \sigma_p ' \Sigma_Y J_{wY} \leq 0 \quad (A6)
\]

\[
m \psi_m = 0 \quad (A7)
\]

\[
\psi_\alpha = \left( \mu_\eta - 1_N \right) J_w + \left( w^2 \sigma_\eta ' \sigma_\eta + w^2 \sigma_\eta ' \Sigma_s ' 0 - w^2 \sigma_\eta ' \sigma_p \delta - \sigma_\eta ' \sigma_p w \right) J_{ww}
\]

\[
+ \sigma_\eta ' \Sigma_Y w J_{wY} \leq 0 \quad (A8)
\]

\[
\alpha \psi_\alpha = 0 \quad (A9)
\]
Equation (A4) is the usual envelope condition. The Kuhn-Tucker conditions (A5) and (A7) account for the non-negativity constraints the consumption process and the real money balances processes, respectively, must satisfy. Similarly, condition (A9) accounts for the impossibility to invest a negative amount of wealth in the real technology. Our assumptions insure that we have an interior solution and therefore we can use (A8) with equality. Using (A8), (A10) and (A11) yields the desired result for the asset demands.

Proof of Proposition 4:

From the optimality conditions (A8) and (A9), we have at equilibrium:

\[ r = \mu + \sigma \left( \sigma \eta - m_{\sigma} \right) \frac{J_{ww}}{J_{w}} + \Sigma \frac{J_{wY}}{J_{w}} \]

(A12)

Applying Ito's lemma to the optimal consumption process written in a feedback form, \( c = c(t, w, Y) \), yields:

\[ dc = c_{\mu} dt + c_{\sigma} dZ \]

(A13)
where \( \sigma_c \equiv \left( w \alpha \sigma_{\eta} - m \sigma_p \right) c_w + \Sigma Y c_Y \) with obvious notations for the first partial derivatives of \( c \) with respect to its arguments (namely, wealth and the \( K \times 1 \) vector of state variables).

Similarly, we have:

\[
dm = m \mu_m dt + m \sigma_m \ 'dZ
\]

where \( \sigma_m \equiv \left( w \alpha \sigma_{\eta} - m \sigma_p \right) m_w + \Sigma Y m_Y \).

Using the envelope conditions (A4) with equality (since we assume that an interior solution exists), one gets:

\[
\frac{J_{ww}}{J_w} = \frac{U_{cc}}{U_c} c_w + \frac{U_{cm}}{U_c} m_w
\]

(A15)

and

\[
\frac{J_{wY}}{J_w} = \frac{U_{cc}}{U_c} c_Y + \frac{U_{cm}}{U_c} m_Y
\]

(A16)

Using (A13) – (A16) yields:

\[
\left( w \alpha \sigma_{\eta} - m \sigma_p \right) \frac{J_{ww}}{J_w} + \Sigma Y \ ' \frac{J_{wY}}{J_w} = \frac{cU_{cc}}{U_c} \sigma_c + \frac{mU_{cm}}{U_c} \sigma_m
\]

(A17)

Substituting into (A12) yields (18). Since \( m = (1 - \alpha)w \), it is easy to see that:

\[
\sigma_m = \sigma_{\eta} - \frac{1 - \alpha}{\alpha} \sigma_M - \frac{1}{1 - \alpha} \sigma_{\alpha}
\]

(A18)
Substituting for (A18) and (13) into (18) yields (19).

Proof of Propositions 5 and 6:

From (A11), it follows that:

\[ R - \left( r + \mu_p - \sigma_p \right) = \sigma_p \left( w \alpha \sigma_\eta - m \sigma_p \right) \frac{J_{ww}}{J_w} + \sigma_p \Sigma_\gamma J_{wY} \frac{J_{wY}}{J_w} \]  
(A19)

Using (A17) leads to equation (22). To obtain equation (21), we substitute for (A19) into (A6) and use (A4). Both (A4) and (A6) are used with a strict equality.

Proof of Proposition 7:

Using the three market clearing conditions and (A10), we have:

\[ \mu_s - I_H r = -\left( w \alpha \Sigma_\eta - m \Sigma_\eta \sigma_p \right) \frac{J_{ww}}{J_w} - \Sigma_\gamma J_{wY} \frac{J_{wY}}{J_w} \]  
(A20)

Using (A17) leads to (23). Equation (24) follows by using (A18) for \( \sigma_m \) and (13) for \( \sigma_c \).

Proof of Proposition 8:

From (A20) we have:

\[ \mu_s - I_H r = -\Sigma_\eta \sigma_w \frac{w J_{ww}}{J_w} - \Sigma_\gamma J_{wY} \frac{J_{wY}}{J_w} \]  
(A21)

For a single asset, we obtain:

\[ \mu_{s_i} - r = -\sigma_{s_i} \left( \sigma_w \frac{w J_{ww}}{J_w} + \Sigma_\gamma J_{wY} \frac{J_{wY}}{J_w} \right) = \sigma_{s_i} \lambda \]  
(A22)
In our Markovian setting, the price of any contingent claim can be written in a feedback form, and therefore \( s_i = s_i(t, w(t), Y(t)) \). Applying Ito's lemma yields:

\[
\sigma_{s_i} = \frac{1}{s_i} \left( \sigma_{s_i w} \sigma_w + \Sigma_Y s_i \right)
\]

Substituting into (A22) yields (27).

**Proof of Proposition 9:**

In the special case of a logarithmic utility function, the standard solution for the investor’s value function is:

\[
J(t, w) = \frac{1}{\rho} e^{-\rho t} \ln(Aw)
\]

where A could be easily determined (to no effect) using standard calculus and the usual transversality conditions, the investor’s horizon being infinite. Our strategy to prove (34) consists in identifying two expressions for \( R \), one related to the ratio of the marginal utility of real balances over consumption, and the second derived from the Fisher equation.

Equations (33), (A4) and (A24) together imply that:

\[
c = \phi w
\]

Using equations (33), (A25) and the fact that at equilibrium \( m = (1-\alpha)w \), the nominal interest rate is equal to:

\[
R = \frac{U_m}{U_c} = \frac{(1-\phi)\rho}{1-\alpha}
\]

Using equation (33), we have:
\[- \frac{U_{\infty}}{U_c} c = 1 \]  
(A27)

Now, using (A27), the equilibrium inflation risk premium given by equation (22) simplifies to:

\[ R - \mu_P + \sigma_p' \sigma_p - r = -\sigma_p' \sigma_c \]  
(A28)

Using equations (18), (31) and (32), the real interest rate is equal to:

\[ r = \mu_{\eta Y}^\lambda_{\sigma} - \eta_{\sigma} \sigma_c \]  
(A29)

Using equations (14) and (32), we get:

\[ \mu_p = \frac{1}{\alpha} \mu_M \eta^{\lambda_{\sigma}} + \left( -\mu_{\eta Y}^\lambda_{\sigma} + \eta^{2\lambda_{\sigma}} \eta_{\sigma} \right) + \frac{\phi\rho}{1 - \alpha} \left( -\mu_a + \sigma_a' \sigma_a \right) \]
\[ + \frac{1}{1 - \alpha} \eta^{\lambda_{\sigma}} \eta_{\sigma} \sigma_M + \frac{1}{(1 - \alpha)^2} \eta_{\sigma} \sigma_a' \sigma_a - \frac{1}{\alpha} \eta^{\lambda_{\sigma}} \eta_{\sigma} \sigma_M - \frac{1}{1 - \alpha} \eta^{2\lambda_{\sigma}} \sigma_a' \sigma_a \]  
(A30)

Using (15), (31) and (32) gives:

\[ \sigma_p = \frac{1}{\alpha} \eta Y_M(t)\eta^\lambda \sigma_M - \eta(t)\eta^\lambda \eta_{\sigma} + \frac{1}{1 - \alpha} \sigma_\alpha \]  
(A31)

Using (13), (A25), (31) and (32) yields:

\[ \sigma_c = \eta(t)\eta^\lambda \sigma_M - \frac{1 - \alpha}{\alpha} \eta Y_M(t)\eta^\lambda \sigma_M - \sigma_\alpha \]  
(A32)

Finally, using (A29) - (A32) and substituting into (A28) gives:

\[ R = \frac{1}{\alpha} \mu_M \eta^{\lambda_{\sigma}} + \frac{1}{1 - \alpha} \mu_a + \frac{\phi\rho}{1 - \alpha} \eta^{\lambda_{\sigma}} \sigma_a' \sigma_a - \frac{1}{\alpha} \eta^{2\lambda_{\sigma}} \sigma_a' \sigma_a \]  
(A33)

Combining (A26) and (A33) yields:

\[ \frac{(1 - \phi\rho)}{1 - \alpha} = \frac{1}{\alpha} \mu_M \eta^{\lambda_{\sigma}} + \frac{1}{1 - \alpha} \mu_a + \frac{\phi\rho}{1 - \alpha} \eta^{\lambda_{\sigma}} \sigma_a' \sigma_a - \frac{1}{\alpha} \eta^{2\lambda_{\sigma}} \sigma_a' \sigma_a \]  
(A34)
It is therefore clear that \( \alpha \) is a function of the monetary state variable only. Equation (34) thus follows.

Rearranging terms in (A34) now gives:

\[
\left( \rho + \mu_M Y_M^{2 \lambda_1} - Y_M^{2 \lambda_1} \sigma_M \sigma_M \right) f(Y_M) + \left( -\mu_\alpha + Y_M^{\lambda_4} \sigma_\alpha \sigma_M \right) f(Y_M) = \phi \rho + \mu_M Y_M^{\lambda_3} - Y_M^{2 \lambda_4} \sigma_M \sigma_M
\]  

(A35)

Applying Ito’s lemma to (34) and using (32), we obtain:

\[
\frac{df}{f} = \left( f' \mu_{Y_M} + \frac{1}{2} f'' Y_M \sigma_{Y_M} \sigma_{Y_M} \right) dt + f' \sqrt{Y_M \sigma_{Y_M} \sigma_{Y_M}} dZ
\]  

(A36)

Substituting into (A35) yields:

\[
\left( \rho + \mu_M Y_M^{\lambda_3} - Y_M^{2 \lambda_1} \sigma_M \sigma_M \right) f + \left( Y_M^{\lambda_4} \sqrt{Y_M \sigma_{Y_M} \sigma_{Y_M}} - \mu_{Y_M} Y_M \right) f' = \phi \rho + \mu_M Y_M^{\lambda_3} - Y_M^{2 \lambda_4} \sigma_M \sigma_M
\]  

(A37)

Rearranging terms yields the desired result.

Proof of Proposition 10:

Equation (40) follows from (A25). Using (40) and (34) and substituting into (11) and (12) yields the expected growth rate of real wealth:

\[
\mu_w = -\frac{1-f}{f} \mu_{Y_M} - \frac{f'}{f} \mu_{Y_M} Y_M - \frac{1}{2} f'' Y_M \sigma_{Y_M} \sigma_{Y_M} + \mu_Y Y_M - \frac{\phi \rho}{f}
\]

\[
+ \left( \frac{f'}{f} \right)^2 Y_M \sigma_{Y_M} \sigma_{Y_M} - \frac{f'}{f} Y_M^{\lambda_4} \sqrt{Y_M \sigma_{Y_M} \sigma_{Y_M}} \sigma_{Y_M} + \frac{2 - f f'}{f} Y_M^{\lambda_4} \sqrt{Y_M \sigma_{Y_M} \sigma_{Y_M}} \sigma_{Y_M}
\]

(A38)

\[
- \frac{1-f}{f} \mu_{\eta_M} Y_M^{\lambda_2} \sigma_{\eta_M} \sigma_M + \frac{1-f}{f^2} Y_M^{2 \lambda_4} \sigma_M \sigma_M
\]

The volatility of this growth rate is given by:
\[ \sigma_w = Y_{\eta}^{\lambda_2} \sigma_{\eta} - \frac{1-f}{f} Y_M^{\lambda_4} \sigma_M - \frac{f'}{f} \sqrt{Y_M} \sigma_{y_M} \]  

(A39)

Equation (37) follows from integrating \( dw/w = \mu_w dt + \sigma_w'dZ \).
References


Duffee, G., 2002, Term premia and interest rate forecasts in affine models, forthcoming in *Journal of Finance*.


