Noise and aggregation of information in competitive rational expectations models

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Abstract

We study a novel set of competitive equilibria in the standard rational expectations setting (Hellwig (1980)) by taking as a parameter the size of the noise trader demand and assuming that its variance is proportional to $N^\beta$, where $N$ denotes the number of agents in the economy and $\beta \in [0, 2]$. We show that when agents’ information acquisition decisions are endogenous, the limiting competitive equilibrium is well-defined and leads to non-trivial information acquisition and partially revealing prices for all $\beta \in [0, 2]$. Three types of limiting equilibria arise as $N \to \infty$: (1) the standard case (see Verrecchia (1982)) is recovered setting $\beta = 2$, where a positive fraction of the traders becomes informed; (2) when $\beta \in (0, 2)$, the number of informed traders grows as $N^{\beta/2}$, so the fraction of informed traders is negligible in the limit; (3) when $\beta = 0$, the number of informed traders is independent of $N$. The paper provides closed-form solutions for all limiting equilibria, and discusses their different implications.

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1 Introduction

Starting with the seminal work of Grossman (1976) and Grossman and Stiglitz (1980), the informational content of prices in competitive market settings has been a subject of significant interest in economics as well as in different areas of applied research. In finance, rational expectations models have been used to study markets for information (Admati and Pfleiderer (1986), Admati and Pfleiderer (1990)), derivatives (Brennan and Cao (1996), Cao (1999)), insider trading (Ausubel (1990a), Bushman and Indjejikian (1995)), security design problems (Duffie and Rahi (1995), Demange and Laroque (1995), Rahi (1996)), and the dynamics of asset prices and volume (Campbell, Grossman, and Wang (1993), Wang (1993)), among other topics. In accounting, many issues around disclosure and compensation have been studied within the standard rational expectations paradigm (see, for example, Diamond and Verrecchia (1982), Diamond (1985), Grundy and McNichols (1989), Banker and Datar (1989), Bushman and Indjejikian (1993), Kim and Verrecchia (1991) and the references in Verrecchia (2001)).

The purpose of this paper is to contribute to the foundations of this literature by generalizing the most widely used theoretical framework with endogeneous information acquisition, the rational expectations model of Hellwig (1980) and Verrecchia (1982). In particular, we study how the amount of noise\(^1\) in the economy affects the rational expectations equilibria. Modelling aggregate random noise variance as proportional to \(N^\beta\), for \(\beta \in [0, 2]\), where \(N\) is the number of traders in the competitive economy, we study limiting equilibria by letting \(N \uparrow \infty\). The case \(\beta = 2\) yields the standard model (Hellwig (1980), Verrecchia (1982)), in which, under some parameter restrictions, a positive fraction of traders becomes informed in equilibrium. In contrast, when \(\beta \in (0, 2)\) we show that the number of informed traders is arbitrarily large as \(N \uparrow \infty\), namely of order \(N^{\beta/2}\), but nevertheless of negligible size as compared to \(N\). In that case the set of informed traders, although uncountable, is of Lebesgue measure zero.\(^2\)

Models with \(\beta = 2\) and \(\beta \in (0, 2)\) share some important features. In particular, prices in all \(\beta > 0\) models are strictly partially revealing.\(^3\) In addition prices, in the limit, depend only on the final payoff of the risky asset and on a variable that parameterizes the risky aggregate

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\(^1\)See Black (1986) for a discussion of the role that noise plays in a broad range of economic models.

\(^2\)It may be more natural to think in terms of a large but finite \(N\): the limiting economy with \(\beta \in (0, 2)\) is one where the number of informed traders is large, although small compared to the size of the economy as measured by \(N\). The fact that the number of informed agents is arbitrarily large justifies the competitive assumption as in the Hellwig (1980) setting.

\(^3\)By this we mean that the conditional variance of the payoff of the risky asset \(X\) given prices is strictly greater than zero and strictly less than the unconditional variance of \(X\).
supply. Therefore, in a limiting economy with $\beta \in (0, 2)$ prices aggregate information just like in the standard model (Hellwig (1980)): individual signals do not show up in the price function. We call such aggregation perfect information aggregation. In the case where the amount of noise does not depend on $N$, i.e. the $\beta = 0$ case, the price function does depend on the individual agents' signals. We call such aggregation imperfect information aggregation.

However, there are some important differences between $\beta = 2$ and $\beta \in (0, 2)$ models. In particular, the two sets of models have different functional forms and several properties of the rational expectations equilibrium are affected. In contrast to the $\beta = 2$ model, where it is possible for all agents to become informed, when $\beta \in (0, 2)$ some agents always choose to stay uninformed no matter how good the information acquisition technology is. In particular, in that case only the most risk-tolerant agents may decide to become informed, while less risk-tolerant agents never do.\(^4\) The paper discusses several other important differences in equilibrium prices and price revelation between the models with $\beta = 2$ and $\beta < 2$ in more detail.

In the literature there are very few papers that deviate from the $\beta = 2$ assumption. One of them is Diamond and Verrecchia (1981). There, the aggregate supply is the sum of $N$ i.i.d. endowment shocks with finite variance. In that case prices do become fully revealing as $N \uparrow \infty$ if a positive fraction of traders has private information. One might think, therefore, that any model that violates the above assumption cannot have partially revealing equilibrium in the large $N$ limit. We show below that such a conjecture is not true. Even if, like in Diamond and Verrecchia (1981),\(^5\) in the limiting economy prices do become fully revealing when an exogenously fixed positive fraction of traders is informed, partial revelation is restored if the fraction of informed traders is determined endogenously: in such models, as $N$ grows large the fraction of informed traders is reduced in size in such a way as to prevent full information revelation.

We analyze a REE model with $N$ traders, where $m^* \leq N$ of them decide to acquire costly private information. The information is acquired at an ex-ante stage and, therefore, the number of informed agents is determined endogenously in our model. In order to study the set

\(^4\)Assuming that the agent population has risk-aversion parameters distributed on a support $[\tau_{\min}, \tau_{\max}]$, in the standard case ($\beta = 2$, Verrecchia (1982)) the informed agents belong to an interval $[\tau_{\min}, \bar{\tau}]$, where $\bar{\tau} \leq \tau_{\max}$. In contrast, when $\beta \in (0, 2)$ only those agents with risk aversion $\tau_{\min}$ become informed.

\(^5\)Diamond and Verrecchia (1981) assume that all agents have private information, and do not study the ex-ante information acquisition problem. Nevertheless, it is easy to see from their equilibrium expressions that prices would perfectly reveal the payoff from the risky asset as $N \uparrow \infty$. 

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of competitive equilibria we first solve for the equilibrium assuming that \( N \) is finite. We then characterize the limiting equilibrium that obtains by letting \( N \uparrow \infty \). Our constructive proofs thus link explicitly the finite-agent models to the limiting economies. Studying the limiting economy allows us to arrive at tractable close-form expressions.

In the main body of this paper we study an economy where traders have CARA preferences with the same risk-aversion parameter. There is one risky and one riskless asset in the economy. The signals that the agents receive are assumed to be i.i.d. conditional on the risky asset payoff. We commence by assuming that the aggregate supply of the risky asset is a random variable independent of agents’ endowments or any other variable of the model. We refer to this as the aggregate noise trader demand (see Kyle (1985) among many others). We further assume that the variance of the aggregate noise is proportional to \( N^\beta \), where \( 0 \leq \beta \leq 2 \). The main result of this paper is to establish that all information acquisition equilibria in that case have limiting equilibrium prices that are partially revealing. In particular, if the equilibrium number of informed agents is represented by the leading term in the large \( N \) asymptotic expansion \( m^* \approx \lambda_\alpha N^{\alpha/2} \), we show that the limiting equilibrium number of informed agents must satisfy \( \alpha = \beta \), i.e. that the rate of growth of the number of informed agents is the same as the rate of growth of the standard deviation of the aggregate supply.

We extend the basic symmetric model in a couple of directions. First, we consider the case of heterogeneous agents. When agents vary in their risk-tolerance, only the most risk-tolerant ones may decide to become informed when \( \beta \in (0,2) \). In another extension, we consider a model in which each agent’s endowment is subject to a random supply shock (see Diamond and Verrecchia (1981)). While the finite \( N \) model in that case looks quite different from the \( \beta = 1 \) model (which serves as a canonical example in this paper), we demonstrate that the corresponding limiting economies coincide. In this way, our paper provides a rationale for the use of models with noise traders, showing that in the limiting economies the equilibrium is the same as in a model where agents face the aggregate residual endowment risk.\(^6\)

The early papers by Hellwig (1980), Diamond and Verrecchia (1981) and Verrecchia (1982) are the closest to our work, both in terms of the motivation and the model setup. The literature on aggregation of information in settings other than financial markets is also related (e.g. Wilson (1977), Milgrom (1979)). There are few papers that consider the endogenous

\(^6\)The early literature (e.g. Verrecchia (1982), Admati (1985)) touch upon this issue informally for the case \( \beta = 2 \).
acquisition of information in such settings: Matthews (1984) does so in the case of auctions, and Vives (1988) studies the aggregation of information in a Cournot-type product market. Although it is difficult, if not impossible, to precisely map the elements of noise discussed in our paper with those of this literature, there are some interesting similarities. In particular, in the oligopoly literature it is typical to model inverse demand functions on a per-capita basis. Whether a change in maintained assumptions on noise factors would have similar effects there to the ones outlined above, and whether such changes would be sound from the economic standpoint, seem to be an interesting route to pursue elsewhere. The paper also contributes to the literature on partially revealing REE (see Ausubel (1990b) and DeMarzo and Skiadas (1998) in addition to the references cited above) by providing a whole new class of models that generates this type of equilibria and that is tractable in closed form.

The structure of the paper is as follows. In section 2 we present our main model which is based on the assumptions of agents’ homogeneity and the existence of some exogenous random aggregate supply (noise traders). Section 3 is the central section of the paper and contains its main results (Propositions 1 to 3). Section 4 relaxes the assumption of agents’ homogeneity by considering, in turn, agents that differ in risk aversion (Proposition 4) and agents that are faced with individual endowment shocks (Proposition 5). Section 5 concludes and presents suggestions for future research. Proofs of the propositions can be found in the Appendix.

2 The model

2.1 Preferences and assets

Consider an economy with $N$ agents. In the bulk of the text we assume that the agents are, ex-ante, identical (we relax this assumption in section 4). We further assume that each agent has CARA preferences with risk-aversion $\tau$, i.e. given a final payoff $W_i$, each agent $i$ derives the expected utility $E[u(W_i)] = E[-\exp(-\tau W_i)]$.

There is a market for a risky asset that has a random final payoff $X \in \mathbb{R}$. After an information acquisition decision process, described below, a subset of agents becomes informed by observing a signal of the form $Y_i = X + \epsilon_i$. We denote by $m$ and $n$ the number of informed

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7Our paper also complements the literature on the aggregation of information in auction environments by analyzing the aggregation of information in different types of market setting. For some recent work in that area see Pesendorfer and Swinkels (2000) and Kremer (2002).
and uninformed agents, respectively. It follows that \( n = N - m \). We let \( \theta_i \), for \( i = 1, \ldots, N \), denote the trading strategy of agent \( i \) (i.e. the number of shares of the risky asset that agent \( i \) acquires). Without loss of generality we can label the informed agents with the subscripts \( 1, \ldots, m \), and the uninformed with the subscripts \( m + 1, \ldots, N \). With this notation, the final wealth for an agent of type \( i \) is given by \( W_i = \theta_i(X - P_s) \).\(^8\) Whenever there is no risk of confusion, trading strategy of a typical informed agent will be denoted by \( \theta_I \), whereas \( \theta_U \) will denote the trading strategy corresponding to an uninformed.

As usual in this type of rational expectations models, we assume that the supply of the risky asset \( Z \) is random (we justify this assumption below). This variable is the main driver in preventing private information to be revealed perfectly to other market participants and plays a significant role in each agent’s information acquisition decision process.

All random variables in the model are defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). It is assumed that the normally distributed random vector \((X, (\epsilon_i), Z)\) has zero mean, and diagonal variance-covariance matrix with corresponding elements \( \sigma_X^2 \), \( \sigma_{\epsilon}^2 \) and \( \sigma_Z^2 \). We will use \( \mathcal{F}_i \) to denote the information set at the time of trading of an agent of \( i \) (for \( i = 1, \ldots, N \)). Note that

\[
\mathcal{F}_i = \begin{cases} 
\sigma(Y_i, P_s) & i = 1, \ldots, m \\
\sigma(P_s) & i = m + 1, \ldots, N;
\end{cases}
\]

where \( \sigma(X) \) denotes the \( \sigma \)-algebra generated by a random variable \( X \).

### 2.2 Definition of equilibrium

A rational expectations equilibrium in this model is defined by a set of trading strategies \( \theta_i \) and a price function \( P_s : \Omega \rightarrow \mathbb{R} \) such that:

1. The trading strategies of all players are optimal, i.e.

\[
\theta_i \in \arg \max_{\theta} \mathbb{E}[u(W_i)|\mathcal{F}_i] \quad i = 1, \ldots, N.
\]

\(^8\)We normalize here, as is customary in the literature, the agents’ initial wealth and the risk-free rate to zero. In addition, there is no borrowing or lending constraints imposed on the agents. These assumptions are innocuous since the model contains only one period of trading.
Markets clear:

\[ \sum_{i=1}^{N} \theta_i = Z. \]  

(1)

As is customary in the REE literature, we conjecture that the equilibrium price is linear in the signals and random supply, namely that

\[ P_s = \sum_{i=1}^{m} b_i Y_i - dZ. \]  

(2)

At the information gathering stage each agent can acquire a signal \( Y_i \) at the cost (measured in units of account) of \( c > 0 \). Therefore, upon acquisition, an agent’s expected utility reads \( \mathbb{E}[u(W_i - c)] \). It should be noted here that this expectation is unconditional, i.e. that it is taken before the signals are realized. In addition, it presumes that the agents anticipate the rational expectations equilibrium price given in (2). We determine the equilibrium number of informed agents, \( m^* \), by equating the ex-ante expected utilities of informed and uninformed agents. This is equivalent to solving for the Nash equilibrium in the non-cooperative game played by the agents, in which their action space is whether to acquire or not to acquire information. An important output of the model is the fraction of informed agents that we denote by \( \lambda^* = m^*/N \).

Note that the model thus far follows closely follows the setup in Verrecchia (1982) with one small difference: instead of assuming that an agent receives a signal with \( \epsilon_i \sim N(0, 1/p) \) at the cost of \( c(p) \) (for \( p \in R_+ \)), we assume, for simplicity, that \( c(p) = c \) for signals with \( p \in [0, 1/\sigma^2_{\epsilon}] \), and \( c(p) = \infty \) for \( p > 1/\sigma^2_{\epsilon} \).\(^9\) This assumption considerably simplifies the algebra but, at the same time, still allows us to capture most of the economically interesting insights.

2.3 Noise

The focus of this paper is on the behavior of asset prices in competitive economies, which we characterize by letting \( N \) tend to infinity in the finite-agent economies. In particular, we are interested in the impact that different assumptions on the distribution of the aggregate random supply \( Z \) have on price formation process and information acquisition decisions. We parametrize noise by letting \( \sigma^2_Z = N^\beta \Sigma^2_Z \), with \( \beta \in [0, 2] \). The case of \( \beta = 2 \) has been extensively studied elsewhere (see, for example, Hellwig (1980), Verrecchia (1982), Admati

\(^9\)See Diamond (1985) for the analysis when \( \beta = 2 \).
(1985), and Diamond (1985, among others). One way to justify such an assumption is to assume that $Z$ is the sum of $N$ random independent shocks to each individual’s endowment of the risky asset, such that each individual shock $Z_i$ is distributed as $N(0, N^2 \Sigma^2_Z)$ (see Verrecchia (1982)). In that case $Z = \sum_{i=1}^{N} Z_i$ would, indeed, satisfy $\sigma^2_Z = N^2 \Sigma^2_Z$. Note that when $\beta = 2$ the variance of each individual agent’s shock has to be scaled by $N$ (since $Z_i$ are i.i.d.) if we were to assure that the variance of the per-capita random supply $Z/N$ is constant and prices do not fully reveal information in the limit when $N \uparrow \infty$ (see Hellwig (1980), for example).

It seems equally interesting (but not much explored in the literature) to consider the case where $Z_i$ is distributed as $N(0, \Sigma^2_Z)$, i.e. where total random supply is the sum of $N$ i.i.d. shocks with a fixed variance, which is along the lines of Diamond and Verrecchia (1981). In our notation such model corresponds to $\beta = 1$. Note that in that case the per-capita random supply shrinks down to zero as $N^{-1/2}$ when $N$ is assumed large. This feature plays an important role in the transmission of information through asset prices, since on a per-capita basis there is almost no noise in the price function. On the other hand, as mentioned in the introduction, if the equilibrium fraction of informed traders stays constant as $N \uparrow \infty$ prices would become fully-revealing in the limit. As a matter of fact, one could make different assumptions on correlations and variances of the individual endowment shocks $Z_i$ that span the whole spectrum $\beta \in (0, 2]$, the cases $\beta = 2$ and $\beta = 1$ corresponding to perfect correlation and independence with constant variance for the shocks. The case $\beta = 0$ can be given a similar interpretation, but now the amount of noise in the market, measured by $\sigma^2_Z$, is independent of the number of agents $N$.

In Section 4.2 we model the aggregate endowment as stemming from endowment shocks that each agent receives prior to the trading round. The main task there is to establish an equivalence between models where aggregate supply is modelled as a “black-box,” versus models where aggregate supply is explicitly modelled. For now, however, we simply assume that the aggregate supply $Z$ is independent of all other random variables in the model, and

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10In contrast to this paper, Diamond and Verrecchia (1981) study a model where the information possessed by the agents is exogenously fixed.

11If all agents knew the value of the terminal payoff $X$ the stock price would be $X$. In contrast, if no agent had any private information about $X$ the stock price would be $0$ (its unconditional mean). In the latter case, no risk-premium would be demanded by the agents, since as $N \uparrow \infty$ each agent’s holdings of the risky asset would become negligible.

12As a matter of fact the case $\beta = 2$ can be obtained assuming that the signals are imperfectly correlated, but that there is non-trivial systemic risk, e.g. assuming that each $Z_i$ is $N(0, \Sigma^2_Z)$ and has a correlation $\rho$ with all other $N - 1$ endowment shocks (it is easy to see that under this set of assumptions $\sigma^2_Z \approx N^2 \rho \sigma^2$).
that it satisfies $\sigma^2_n = N^\beta \Sigma^2_n$, where $\Sigma^2_n > 0$. For future reference, we denote an economy characterized by a parameter $\beta$ and $N$ agents by $E_N(\beta)$ and the (endogenous) number of informed agents as a function of $N$ and $\beta$, which we denote by $m^*_N(\beta)$.

### 2.4 Limiting equilibria

Because we would like to study information acquisition in a competitive economy, we need to understand properties of the equilibria in the economies $E_N(\beta)$ in the large $N$ limit. We do so by first finding closed-form solutions to finite $N$ equilibria, and then letting $N$ go to infinity. We denote the corresponding limiting economy by $E(\beta)$ and the corresponding endogenous number of informed agents by $m^*(\beta)$. Assuming that the number of informed agents $m^*$ is an analytical function of $N$, when the number of agents in the economy is large, the number of informed agents can be expressed as $m^* = \lambda_\alpha N^{\alpha/2} + o(N^{\alpha/2})$, for some non-negative $\alpha$. Here, the second term in the equation are terms in the asymptotic expansion of $m^*$ that are negligible in comparison with $N^{\alpha/2}$ and that can be dropped from the analysis for our purposes. Our task is, then, to find pairs of $(\alpha, \lambda_\alpha)$ for which a limiting equilibrium in the information acquisition stage exists. In what follows, it is useful to define $\lambda^*_\alpha(N) \equiv N^{\alpha/2} m^*_N(\beta)$. The problem of finding the limiting information acquisition equilibrium reduces, in this way, to finding $\alpha$ such that $\lambda^*_\alpha(N)$ has a well-defined positive limit when $N \uparrow \infty$.

The following formalizes the notion of a limiting equilibrium.

**Definition 1.** We say that an economy $E_N(\beta)$ has a limiting information acquisition equilibrium if there exist $\alpha \in [0, 2]$ and $\lambda^*_\alpha \in \mathbb{R}_+$ such that

$$
\lim_{N \to \infty} \lambda^*_\alpha(N) = \lim_{N \to \infty} m^*_N(\beta) N^{-\alpha/2} = \lambda^*_\alpha.
$$

(3)

Note that in this definition we require that $\lambda_\alpha > 0$, i.e. that the equilibrium set of informed agents is not empty. Note, in addition, that in the case $\alpha = 2$ we need to impose one further constraint, namely $\lambda_2 \leq 1$, since the set of informed agents $m$ cannot, in economic terms, be larger than the total number of agents $N$. In cases where $\alpha < 2$ this second constraint is not necessary, since $m$ is always smaller in magnitude than $N$.

The restriction to $\lambda_\alpha > 0$ implies that an information technology needs to be sufficiently “good” for agents to be willing to purchase information. The following formalizes that require-
Definition 2. We say that an economy $\mathcal{E}_N(\beta)$ has a non-trivial information acquisition technology if

$$C(\tau) \equiv e^{2\tau c} - 1 < \frac{\sigma_X^2}{\sigma^2}.$$  

(4)

Of particular interest is the informational content of asset prices as $N \uparrow \infty$. We use $P_s^N$ to denote the equilibrium price function of the risky asset in an economy with $N$ agents. The following definition is commonly used in the literature.

Definition 3. Let $\mathcal{E}(\beta)$ have a limiting information acquisition equilibrium. We say that $\mathcal{E}(\beta)$ has a partially revealing REE if

$$\lim_{N \to \infty} \text{var}(X|P_s^N) \in (0, \sigma_X^2);$$

i.e., if prices neither fully reveal information nor are they completely uninformative.

One technical caveat is in order. In the next section we will discuss convergence properties of the sequence of prices $P_s^N$. The reader should bear in mind that all convergence statements there are both a.s. as well as in $L^2$, although, for brevity, we omit these qualifiers in what follows.

3 Limiting equilibria

Before we analyze the information acquisition problem, we need to establish the fact that in order for prices to be partially revealing we need that, in equilibrium, $\alpha = \beta$. In particular, we show next that if $\mathcal{E}_N(\beta)$ has a limiting information acquisition equilibrium $(\alpha, \lambda^*_\alpha)$, then prices will be asymptotically partially revealing if and only if $\alpha = \beta$.

Proposition 1. Assume that $\mathcal{E}_N(\beta)$ has a limiting information acquisition. A necessary and sufficient condition for prices to be asymptotically partially revealing is that $\alpha = \beta$. If $\alpha = \beta > 0$, then we have that for some $\hat{a}_\beta, \hat{d}_\beta \in \mathbb{R}_+$

$$\lim_{N \to \infty} P_s^N = \hat{a}_\beta X - \hat{d}_\beta \hat{Z}_\beta;$$

(5)

13The next assumption coincides with the condition for which a strictly positive fraction of traders becomes informed in the $\beta = 2$ case (see Diamond (1985)).
where \( \hat{Z}_\beta \equiv \lim_{N \to \infty} N^{-\beta/2} Z \).

The above Proposition shows that in order to have prices be partially revealing in the limit, it is necessary for \( m^*_N(\beta) \) to grow at the rate \( N^{\beta/2} \). Moreover, the price of the risky asset, in the limit, only depends on the vector \((X, \hat{Z}_\beta)\), i.e. individual agents’ signals do not affect prices, just as in Hellwig (1980). In order to gain some intuition, it is useful to re-write the pricing function in (2) by letting \( m^* = \lambda_{\alpha} N^{\alpha/2} \):

\[
P_s = dN^{\beta/2} \left[ \lambda_{\alpha} r N^{(\alpha-\beta)/2} X + N^{(\alpha/2-\beta)/2} \hat{e} - \hat{Z}_\beta \right];
\]

where \( r \equiv b_i/d \), and \( \hat{e} \sim N(0, \lambda_{\alpha} r^2 \sigma^2 \epsilon) \).14 This decomposition represents the price function in terms of three random variables whose distributions are independent of \( N \), namely \( X, \hat{e} \) and \( \hat{Z}_\beta \), scaling them by the appropriate powers of \( N \).

From the above characterization, as long as the relative coefficient on the \( Y_i \) and \( Z \) terms (measured by the quantity \( r \)) has a finite and positive limit, one may be tempted to think that Proposition 1 is almost immediate. However, note that in order to prove the Proposition it is actually necessary to solve for the equilibrium prices in the finite \( N \) economy since the \( r \) variable in (6) is endogenously determined. Once we establish that \( \lim_{N \to \infty} r > 0 \), the basic intuition can be gathered simply by inspection of (6). The conditional variance of \( X \), given price, depends on the relative weights on the random variables \( X, \hat{e} \) and \( \hat{Z}_\beta \). If \( \alpha < \beta \) then the dominant weight is the one on \( \hat{Z} \), i.e., in the limit as \( N \uparrow \infty \), the price conveys no information about \( X \). When \( \alpha > \beta \) the opposite occurs: in the limit as \( N \uparrow \infty \) prices perfectly reveal \( X \).

This implies, among other things, that prices become fully revealing as \( N \uparrow \infty \) if the fraction of informed agents \( \lambda \) is (exogenously) kept constant when \( \beta = 1 \). This is to be expected: the main motivation for choosing \( \beta = 2 \), as in the standard model, is precisely to assure that prices stay partially revealing in the limit and a positive fraction of informed agents trades in the financial market. Note that such justification for the standard \( \beta = 2 \) model neglects the fact that information decisions of traders are actually endogenous. Moreover, exogenously specifying \( \beta = 2 \) eliminates potentially interesting equilibrium in which the set of informed traders may not be a set with positive Lebesgue measure. The purpose of this paper is to show that indeed, once we solve explicitly the endogenous information acquisition problem,
the limiting economies will possess partially revealing prices for all \( \beta \in [0, 2] \). This confirms the hypothesis against fully revealing equilibria discussed in the early literature on rational expectations models (see Diamond and Verrecchia (1981) for example), as well as highlights other equilibria that share this feature besides the \( \beta = 2 \) case. Moreover, note that as \( N \uparrow \infty \) the estimation errors vanish from the pricing formula as long as \( \alpha < 2 \beta \), which is always the case in asymptotically partially revealing equilibria as long as \( \beta > 0 \). Thus, prices in this case aggregate diverse pieces of information perfectly (in the terminology introduced above) and idiosyncratic noise from the signal realizations does not show up in price, just as in the standard \( \beta = 2 \) model (Hellwig (1980), Verrecchia (1982)).

The next Proposition is the central result from this paper.

**Proposition 2.** Consider an economy \( E_N(\beta) \) with a non-trivial information acquisition technology and \( \beta \in (0, 2] \). In a limiting information acquisition equilibrium asset prices are asymptotically partially revealing and the price function satisfies

\[
\lim_{N \to \infty} P^N_s = \hat{a}_\beta X - \hat{d}_\beta \hat{Z}_\beta;
\]

for some \( \hat{a}_\beta, \hat{d}_\beta > 0 \). The number of informed traders satisfies

\[
\lim_{N \to \infty} m^*(N) N^{-\beta/2} = \lambda_\beta;
\]

for some \( \lambda_\beta > 0 \). In particular:

(i) If \( \beta = 2 \) we have that

\[
\hat{a}_2 = \frac{\lambda_2^*}{\tau \sigma_\epsilon^2} \hat{d}_2; \quad \hat{d}_2 = \frac{1 + \frac{\lambda_2^* r_2}{\tau \Sigma_Z^2} + \frac{\lambda_2^* r_2^2}{2 \Sigma_Z^2}}{\lambda_2^* r_2 + \frac{1}{\tau} \left( \frac{\lambda_2^* r_2}{\Sigma_Z^2} + \frac{1}{\sigma_X^2} \right)}; \quad r_2 = \frac{1}{\tau \sigma_\epsilon^2}.
\]

If \( \frac{1/\sigma_X^2 + r_2/\Sigma_Z^2}{1/\sigma_X^2} < C \) then

\[
\lambda_2^* = \tau \Sigma_Z \sigma_\epsilon \sqrt{\frac{1}{C} - \frac{\sigma_\epsilon^2}{\sigma_X^2}};
\]

otherwise \( \lambda_2^* = 1 \).
(ii) If $0 < \beta < 2$ the price coefficients are given by

$$
\hat{a}_\beta = \frac{\lambda_2^*}{\tau \sigma_c^2} \hat{d}_\beta; \quad \hat{d}_\beta = \frac{\lambda_2^* r_\beta / \Sigma_Z^2}{1/\sigma_X^2 + (\lambda_2^* r_\beta)^2}; \quad r_\beta = \frac{1}{\tau \sigma_c^2}.
$$

Moreover, $\lambda_2^* = \lambda_2^*$, as given in (10).

Proposition 2 establishes the fact that in the limit as $N \uparrow \infty$ all equilibria in which the information acquisition technology is non-trivial will lead to limiting prices that are partially revealing. Note that this is rather intuitive: if prices were fully revealing there obviously would be no reasons for agents to gather any information. But the Proposition 2 also establishes that if the information technology (summarized by $c$ and $\sigma_c^2$) is non-trivial, it also cannot be the case that prices are completely uninformative. In particular, all informed agents put non-trivial weights on their private information and the risky asset price, and the price acts as an aggregator of the private information possessed by agents, without being fully revealing.

It is interesting to analyze the limiting trading strategies. Consider first the limiting economy with $\beta = 2$. The equilibrium trading strategy for the informed is given by $\theta_I = r_2 Y_i + q_2 P_s$, and for the uninformed by $\theta_U = w_2 P_s$, where $r_2$, $q_2$ and $w_2$ all have finite limits. One can show that in this limiting economy informed agents take the same position as uninformed, plus they add an amount $r$ times the signal and subtract the same amount times the price from the uninformed trading strategy. Informed agents put non-trivial weights both on the risky asset price (which is partially revealing) and on their private information. Both informed and uninformed take non-trivial positions in the limit when $N \uparrow \infty$.

Now consider the trading behavior when $\beta = 1$. The equilibrium trading strategy for the informed is given by $\theta_I = r_1 Y_i + q_1 P_s$, and for the uninformed by $\theta_U = w_1 P_s$, where $\lim_{N \to \infty} q_1 = -r_1$ and $\lim_{N \to \infty} w_1 = 0$. In this limit each uninformed agent holds a negligible amount of the risky asset, whereas the informed do take non-trivial positions based on their private information and the risky asset price. Note that the fact that individual trades are negligible does not mean that the uninformed do not play a role in equilibrium price formation. In fact, it can be shown that the aggregate trade by the uninformed is of the same magnitude as the trade by the informed (of the order $N^{\beta/2}$).

Note that the price coefficients in the $\beta = 2$ and the $\beta \in (0, 2)$ cases actually have different functional forms, as shown in Proposition 2. One the other hand, the limiting $\lambda_2^*$, which
summarize the information acquisition decision by the agents, have the same functional form for all $\beta \in (0, 2]$ as long as $\lambda_2^2 \leq 1$. As we will show in section 4.1, this depends critically on the assumption of agent homogeneity. The equilibrium expected utilities in the $\beta = 2$ and $\beta \in (0, 2)$ cases are actually rather different. In the standard case, both uninformed and informed earn some rents in equilibrium, which depend on the informativeness of prices. When $\beta < 2$, on the other hand, as $N \uparrow \infty$ the uninformed actually earn no rent, since their trading strategy converges to zero. In other words, per-capita random supply is, in that case, converging to zero so their equilibrium share of the final risky payoff converges to zero as well. Nevertheless, the expected utility of the informed does depend on the informativeness of prices relative to the signal. The way this enters into informed expected utility turns out to have the same functional form as the relative utilities of informed and uninformed agents in the $\beta = 2$ case. An important driver of this equivalence is that the signal-to-noise ratio in the price function depends on precisely the same parameter values, the number of informed agents times the aggressiveness of their trading strategy, which is given by $1/(\tau \sigma_e^2)$ in both cases.

In the limiting economy with $\beta \in (0, 2)$ there is an infinite number of informed agents, although a negligible number when compared to the whole population of traders. This brings out another property of this limiting economy: there will always be some uninformed agents left in the market. This contrasts with the case $\beta = 2$, where it is possible to have a corner solution where all agents become informed.

Table 1 contains a comparison of the equilibrium without information ($\lambda_\beta = 0$) with that in which agents endogenously acquire some information. Consider what would happen in these economies if there were no information acquisition. When per-capita supply is non-trivial ($\beta = 2$) agents will demand a risk-premium $\hat{d}_2 \hat{Z}$ for holding this risky asset. Once it becomes possible to acquire information, the equilibrium price includes a new term that depends on the final payoff of the risky asset, thereby making risky asset prices partially revealing (and at the same time changing the risk-premium demanded by agents). Case $\beta \in (0, 2)$ has an even more dramatic change from the no-information equilibrium to the one where there exists an information technology that agents can invest in. In the former case the price of the risky asset is simply zero (i.e. equal to its unconditional expected value). Once agents can invest in information-gathering, the price of the risky asset becomes a non-trivial random variable: $X$ shows up in the price function due to the trading by the informed, and a non-trivial risk-premium arises (the $Z$ term). These differences between the two models suggest that measures
such as the release of information by firms (through, say, their accounting statements) may have different impact on the equilibrium properties of asset prices depending on the model under consideration.

For completeness we present the following result.

**Proposition 3.** For $\beta = 0$, as $N \uparrow \infty$ the number of informed agents satisfies $m^*(N) = \lambda_0^*$, i.e. the number of informed agents is a constant independent of $N$.

This result highlights the fact that the $\beta = 0$ case is indeed significantly different from the $\beta > 0$ cases. In particular, since only a finite number of agents becomes informed, individual signals will show up in price, so on top of the usual terms related to $X$ and $Z$ there will be non-trivial estimation errors in the price function, i.e. price aggregates information imperfectly. The Appendix contains the details on the actual equilibrium price coefficients. Not surprisingly, the fact that the price function has the above extra elements makes this particular case more complicated to analyze, as compared to the $\beta \in (0, 2]$ cases. Moreover, the competitive assumption is harder to justify in this case, since agents are never infinitesimal in terms of their effect in price, in contrast to the $\beta \in (0, 2]$ cases.

### 4 Extensions

#### 4.1 Heterogeneous risk-aversion

We consider next the effects of heterogeneity in the risk-aversion of the individual agents, fixing the information acquisition technology. The definition of a rational expectations equilibrium follows as in the previous section. The only caveats are with respect to the equilibrium at the information acquisition stage. We start with a set of $N$ agents, each with CARA preferences, but a potentially different risk-aversion parameter $\tau_i$. Each agent $i$ takes an action from the set $\{A, N\}$, where $A$ denotes to acquire information, and $N$ denotes the action of not purchasing the signal.

A Nash equilibrium (in pure strategies) in the information acquisition stage is defined by two sets of agents

\[ I = \cup_i \{i : \text{agent } i \text{ purchases a signal}\} \]

\[ N = \cup_i \{i : \text{agent } i \text{ does not purchase a signal}\} \]
such that none of the agents in $\mathcal{N}$ desire to purchase signals, and none of the agents in $\mathcal{I}$ desire not to purchase signals, all taking as given that the other players follow their equilibrium strategies. Let the distribution of risk-aversion types as $N \uparrow \infty$ approach a distribution function $\mu(\tau)$, where the support of $\mu$ is $[\tau_{\text{min}}, \tau_{\text{max}}]$.

For any given finite $N$, many different equilibria exist, and multiplicity of such equilibria will be expected. Nevertheless, for $N$ large enough the equilibrium set of players who become informed takes on a particularly simple form, as established in the next Proposition.

**Proposition 4.** In the model with heterogeneous risk-aversion and $\beta \in (0, 2)$, as $N \uparrow \infty$, the only Nash equilibria that survives is one where only agents with risk-aversion parameter $\tau_{\text{min}}$ become informed. The number of such informed traders tends to $\lambda^*_\beta N^{\beta/2}$, where $\lambda^*_\beta$ is given by

$$\lambda^*_\beta = \tau_{\text{min}} \Sigma Z \sigma_\epsilon \sqrt{\frac{1}{C(\tau_{\text{min}})} - \frac{\sigma^2_\epsilon}{\sigma^2_X}}.$$  

(12)

The effects of heterogeneity go in the same direction as in Verrecchia (1982): the more risk-tolerant are more likely to be informed. But the situation with $\beta \in (0, 2)$ is more extreme: only those agents with the lowest risk-aversion in the population become informed, rather than a strict positive subset of the type space. It is rather intuitive that more risk-tolerant agents will become informed: they trade more aggressively, thereby capturing more rents from their information acquisition activities. This standard intuition, together with the fact that the agents who become informed are negligible in size as $N \rightarrow \infty$, yields the conclusion of the Proposition.

Note that the informativeness of asset prices, measured as usual by the variance of $X$ conditional on price, is a simple monotone function of $\hat{a}/\hat{d} = (\Sigma Z / \sigma_\epsilon) \sqrt{\frac{1}{C(\tau_{\text{min}})} - \frac{\sigma^2_\epsilon}{\sigma^2_X}}$, from which it is immediate that the model with $\beta < 2$ will have prices that are more informative than the model with $\beta = 2$. This is rather intuitive given the previous results, since it will be the most risk-tolerant agents those that make price partially revealing through their trades, versus a subset of the type space in the $\beta = 2$ case.

---

15 It is easy to construct an example with two agents with different risk-aversion parameters, in which only one agent becomes informed in equilibrium, but it could be either of the agents.
4.2 Endowment risk

Consider next a version of the model where noise stems from some random endowment shocks to each agent. In particular, prior to trading each agent observes his endowment of the risky asset, which we denote by $Z_i$. The $Z_i$'s are assumed to be i.i.d. Gaussian random variables with zero mean and variance $\Sigma^2_Z$. Aggregate supply of the risky asset is then $Z = \sum_{i=1}^{N} Z_i$, and has a variance proportional to $N$: therefore we will compare this model to the one described in the main body of the paper setting $\beta = 1$.\textsuperscript{16} Using the notation of the previous sections, the final wealth for agent $i$ is given by $W_i = \theta_i(X - P_s) + P_s Z_i$. All agents are assumed to have CARA preferences with risk-aversion parameter $\tau$. The model described this far is a simple generalization of Diamond and Verrecchia (1981), who only study the case $m = N$, i.e. the case where all agents are informed.

We will conjecture, as usual, that price is linear in the random variables $Y_i$ and $Z_i$. Now there is heterogeneity in the random supply: a portion will stem from informed agents, and a portion from the uninformed. Therefore, we will search for price functions of the form

$$P_s = b \sum_{i=1}^{m} Y_i - d_I \sum_{i=1}^{m} Z_i - d_U \sum_{i=m+1}^{N} Z_i. \quad (13)$$

At the ex-ante stage we proceed as before: we determine the equilibrium number of informed agents by equating the expected utilities of the informed and the uninformed.

The next Proposition shows that the limiting behavior of this economy is asymptotically identical to that in Proposition 2 with $\beta = 1$.

**Proposition 5.** The number of informed agents satisfies $\lim_{N \to \infty} m^* N^{-1/2} = \lambda_1^*$, where $\lambda_1^*$ is given in Proposition 2. The price function (13) converges to (7).

It should be noted that the models do differ in the finite-agent case: in the above model with endowment shocks even the uninformed agents have some private information, namely the realization of $Z_i$. Therefore their trading strategy is slightly more complicated than in the model with noise traders, although in the limit they converge. The above Proposition thereby links the models with endowment risk and noise traders in a competitive setting, for the case of i.i.d. endowment shocks.

\textsuperscript{16}Similar statements can be made about the equivalence that we will establish below for $\beta \in (0, 2]$, parametrizing $\Sigma^2_Z$ to depend on $N$.\textsuperscript{17}
5 Conclusion

This paper studies competitive limiting equilibria in a standard rational expectations framework by considering the ex-ante information acquisition decision of the agents. We show that the way noise enters the model has important consequences for equilibrium price formation as well as information acquisition decisions by the agents. Modelling the noise in the economy as proportional to $N^\beta$, where $N$ is the number of agents in the economy, we demonstrate the existence of three well differentiated types of equilibria: the classic $\beta = 2$ case; the case $\beta \in (0,2)$, which has been the main focus of this paper; and the $\beta = 0$ case. The last two cases have the interesting feature that only a small fraction of the total agents becomes informed in equilibrium (in the limit, this is a negligible amount although there are an infinite number of them) but prices nevertheless (partially) reflect their private information.

We demonstrate that some of the properties of the $\beta = 2$ model carry over to the $\beta \in (0,2)$ case: the price of the stock reveals an aggregate of the informed agents’ signals, the individual signals do not show up in price, the signal-to-noise ratio is the same in the symmetric model, and the equilibrium number of traders depends on the same parameter values in the symmetric case. Nevertheless significant differences appear besides the size of the informed population: the actual price coefficients have different functional forms, prices are generally strictly more informative in the $\beta \in (0,2)$ case, and in the model with heterogeneous risk-aversion parameters only the agents with the lowest risk-tolerance become informed. When $\beta = 0$, the number of informed agents is independent of $N$, and while prices still partially reveal the information, the aggregation of information is not perfect. In other words, individual agent’s signals enter the price function.

The paper aims at furthering our understanding of the foundations of the REE models in securities markets. Its results can be used in diverse areas of the applied research. In particular, it would be interesting to investigate the effect of the differences in the equilibrium prices formation for $\beta \in [0,2)$ and $\beta = 2$ models in the analysis of public information releases (disclosure), compensation in REE models, and security design issues, among other topics. No less importantly, one could analyze empirically the relationship between $N$ and $m^*$. This would clarify which of the competitive models discussed above seems to fit different competitive market settings best.
Appendix

We commence by stating two Lemmas which are used throughout the paper. In the next Lemma we take as fixed $m$, the number of informed agents, and solve for the equilibrium price in $E_N(\beta)$.

**Lemma 1.** Let $m \geq 2$. Then, there exists a symmetric equilibrium of the form (2), in which

\[ d = \frac{d_u}{d_l}; \quad b = r \]  \hspace{1cm} (14)

where $r$ is the solution to

\[ r^3 + r \frac{\sigma_Z^2}{(m-1)\sigma^2_{\epsilon}} = \frac{\sigma_Z^2}{\tau(m-1)\sigma^4_{\epsilon}} = 0. \]  \hspace{1cm} (15)

and

\[ d_u = 1 + \frac{nmr\sigma_Z^2}{r\sigma^2_x(mr^2\sigma^2_{\epsilon} + \sigma^2_{Z})} + \frac{m(m-1)r\sigma_Z^2\sigma^2_{\epsilon}}{r\sigma^2_x\sigma^2_\epsilon(\sigma_Z^2 + r^2\sigma^2_x(m-1))} \]

\[ d_l = \frac{n(r^2m(m\sigma^2_x + \sigma^2_{\epsilon}) + \sigma^2_Z)}{r\sigma^2_x(mr^2\sigma^2_{\epsilon} + \sigma^2_{Z})} + \frac{m(\sigma_Z^2(\sigma^2_{\epsilon} + \sigma^2_Z) + \sigma^2_r^2(m-1)(\sigma^2_{\epsilon}m^2 + \sigma^2_{\epsilon}))}{r\sigma^2_x\sigma^2_\epsilon(\sigma_Z^2 + r^2\sigma^2_x(m-1))} \]

The trading strategies of the informed and uninformed take the following form

\[ \theta_I = rY_i + qP_s; \quad \theta_U = wP_s; \]

where

\[ w = \frac{rm}{\tau (mr^2\sigma^2_{\epsilon} + \sigma^2_{Z})} \left( \frac{1}{d} \right) - \frac{m^2r^2\sigma^2_x + mr^2\sigma^2_{\epsilon} + \sigma^2_Z}{r\sigma^2_x(mr^2\sigma^2_{\epsilon} + \sigma^2_{Z})} \]  \hspace{1cm} (16)

\[ q = \frac{(m-1)r}{\tau (r^2\sigma^2_{\epsilon}(m-1) + \sigma^2_{Z})} \left( \frac{1}{d} \right) - \frac{[\sigma^2_Z(\sigma^2_x + \sigma^2_{\epsilon}) + \sigma^2_r^2(m-1)(\sigma^2_{\epsilon}m + \sigma^2_{\epsilon})]}{r\sigma^2_x\sigma^2_\epsilon(r^2\sigma^2_{\epsilon}(m-1) + \sigma^2_{Z})} \]  \hspace{1cm} (17)

**Proof of Lemma 1.** By standard arguments based on the projection theorem it is straightforward to check that the optimal trading strategy by the uninformed agents is given by

\[ \theta_U = \frac{\mathbb{E}[X|P_s] - P_s}{\tau \text{var}(X|P_s)} \]

\footnote{Note that the condition $m = 2$ is innocuous, since we will mostly be interested in interior solutions, i.e. markets where as $N \uparrow \infty$ the number of informed traders will be some positive amount, and ignore for the most part corner solutions.}
\[
\begin{align*}
\tau &= \frac{P_s \left( b_m \sigma_x^2 - b_m^2 \sigma_x^2 - S - d^2 \sigma_Z^2 \right)}{\sigma_x^2 (S + d^2 \sigma_Z^2)} \\
&\equiv wP_s;
\end{align*}
\]

where \( b_m = \sum_{i=1}^{m} b_i \) and \( S = \sum_{i=1}^{m} b_i^2 \sigma_x^2 \).

Similarly, the optimal trading strategy for an informed agent is given by the expression

\[
\theta_I = \frac{\mathbb{E}[X|Y_i, P_s] - P_s}{\tau \text{var}(X|Y_i, P_s)} = \frac{1}{\tau} \left( \frac{(S + d^2 \sigma_Z^2 - b_m b \sigma_x^2)}{\sigma_x^2 (S + d^2 \sigma_Z^2 - b^2 \sigma_x^2)} \right) Y_i + \frac{(\sigma_x^2 \sigma_x^2 (b_m - b) - \kappa)}{\sigma_x^2 \sigma_x^2 (S + d^2 \sigma_Z^2 - b^2 \sigma_x^2)} P_s
\]

\[
\equiv rY_i + qP_s;
\]

where

\[
\kappa = d^2 \sigma_x^2 (\sigma_x^2 + \sigma_x^2 + \sigma_x^2 \sigma_x^2) + \sigma_x^2 \sigma_x^2 (\sigma_x^2 \sigma_x^2 m (m - 1) + (m - 1) \sigma_x^2);
\]

\[
\sigma^2_P = mb^2 (m \sigma_x^2 + \sigma_x^2) + d^2 \sigma_Z^2.
\]

The market clearing condition reads

\[
\sum_{i=1}^{m} (r_i Y_i + q_i P_s) + \sum_{j=1}^{n} w_j P_s = Z
\]

or

\[
r \sum_{i=1}^{m} Y_i - Z = HP_s;
\]

with \( H = -(mq + nw) \), which boils down to two equilibrium conditions

\[
b = rd; \quad 1 = Hd.
\]

Therefore

\[
r = \frac{1}{\tau} \left[ \frac{d^2 \sigma_Z^2}{\sigma_x^2 (d^2 \sigma_Z^2 + (m - 1) b^2 \sigma_x^2)} \right] = \frac{1}{\tau} \left[ \frac{\sigma_Z^2}{\sigma_x^2 (\sigma_Z^2 + (m - 1) r^2 \sigma_x^2)} \right]
\]

so that the equilibrium value for \( r = b/d \) is given by the solution to the cubic equation (15),
that is

\[ r = \left( \frac{\sigma^2_Z}{2 \tau (m-1) \tau^2} \right)^{1/3} \left[ (1 + \xi)^{1/3} + (1 - \xi)^{1/3} \right] \] (18)

with

\[ \xi = \left( 1 + \frac{4 \sigma^2_Z \sigma^2_r \tau^2}{27 (m-1)} \right)^{1/2}. \]

Furthermore

\[ w = \frac{r dm \sigma^2_Z - m^2 r^2 d^2 \sigma^2_Z - mr^2 d^2 \sigma^2_r - d^2 \sigma^2_Z}{\tau \sigma^2_Z d^2 (mr^2 \sigma^2_r + \sigma^2_Z)} \]

\[ q = \frac{(\sigma^2_Z \sigma^2_r (m-1) rd - d^2 [\sigma^2_Z (\sigma^2_r + \sigma^2_r) + \sigma^2_r r^2 (\sigma^2_Z m (m-1) + (m-1) \sigma^2_r)])}{\tau \sigma^2_Z \sigma^2_r d^2 (mr^2 \sigma^2_r + \sigma^2_Z - r^2 \sigma^2_r)} \]

The equilibrium condition for \( d \) is \( 1 = -d (mq + nw) \), which reduces to

\[ d \left[ \frac{n (r^2 m (ma^2_Z + \sigma^2_r) + \sigma^2_Z)}{\tau \sigma^2_Z (mr^2 \sigma^2_r + \sigma^2_Z)} + \frac{m (\sigma^2_Z (\sigma^2_r + \sigma^2_r) + \sigma^2_r r^2 (m-1) (\sigma^2_Z m + \sigma^2_r))}{\tau \sigma^2_Z \sigma^2_r (\sigma^2_Z + r^2 \sigma^2_r (m-1))} \right] = 1 + \frac{nm m \sigma^2_Z}{\tau \sigma^2_Z (mr^2 \sigma^2_r + \sigma^2_Z)} + \frac{m (m-1) r \sigma^2_Z \sigma^2_r}{\tau \sigma^2_Z \sigma^2_r (\sigma^2_Z + r^2 \sigma^2_r (m-1))} \]

This completes the proof. □

The next Lemma characterizes the equilibrium number of agents that become informed.

**Lemma 2.** The equilibrium number of informed traders is given by the condition

\[ \frac{1}{2} \log \left( \frac{|I + \tau \Sigma_I (m^*) A_I (m^*)|}{|I + \tau \Sigma_U (m^*) A_U (m^*)|} \right) = \tau c \] (19)

where

\[ \Sigma_U (m) = \begin{pmatrix} \sigma^2_P & m b \sigma^2_x \\ m b \sigma^2_x & \sigma^2_x \end{pmatrix}; \quad A_U (m) = \begin{pmatrix} -2w & w \\ w & 0 \end{pmatrix} \]

\[ \Sigma_I (m) = \begin{pmatrix} \sigma^2_P & m b \sigma^2_x & b \sigma^2_x \\ m b \sigma^2_x & \sigma^2_x & 0 \\ b \sigma^2_x & 0 & \sigma^2_x \end{pmatrix}; \quad A_I (m) = \begin{pmatrix} -2q & q - r & -r \\ q - r & 2r & r \\ -r & r & 0 \end{pmatrix} ; \]

and \( \sigma^2_P = (mb)^2 \sigma^2_X + mb^2 \sigma^2_r + d^2 \sigma^2_Z \).
Proof of Lemma 2. The informed agents receive final payoff equal to

\[ W_I = \theta_I(X - P_s) = (rY_i + qP_s)(X - P_s) \]

whereas the uninformed receive

\[ W_U = \theta_U(X - P_s) = wP_s(X - P_s) \]

In order to calculate the ex-ante expected utility for these agents, note that we can write

\[ \mathbb{E}[-\exp(-\tau W_U)] = \mathbb{E} \left[ -\exp \left( \frac{1}{2} \tau V^\top A_U V \right) \right] \]

with \( V^\top = (P_s, X) \) and

\[ A_U = \begin{pmatrix} -2w & w \\ w & 0 \end{pmatrix} \]

The above expression is just the expectation of a quadratic function of Gaussian random variables, for which it is well-known that a closed-form exists.\(^\text{18}\) Letting

\[ \Sigma_U \equiv \text{var}(P_s, X) = \begin{pmatrix} \sigma_P^2 & mb \sigma_x^2 \\ mb \sigma_x^2 & \sigma_x^2 \end{pmatrix} \]

we have that

\[ \mathbb{E}[-\exp(-\tau W_U)] = -|I + \tau \Sigma_U A_U|^{-1/2}. \]

Similarly, we can write informed trader’s payoff (without considering the cost he pays for the fee) as

\[ W_I = -qP_s^2 + rX^2 + XP_s(q - r) + rX\epsilon_i - rP_s\epsilon_i \]

so that

\[ \mathbb{E}[-\exp(-\tau W_I)] = \mathbb{E} \left[ -\exp \left( \frac{1}{2} \tau V^\top A_I V \right) \right] \]

\(^{18}\)In particular, if \( X \sim \mathcal{N}(0, C) \), then \( \mathbb{E}[-\exp(X^\top AX)] = -|I - 2CA|^{-1/2} \) as long as \( |I - 2CA| > 0 \) (see for example Marín and Rahi (1999)).
with $V = (P_s, X, \epsilon_i)$ and

$$A_I = \begin{pmatrix} -2q & q - r & -r \\ q - r & 2r & r \\ -r & r & 0 \end{pmatrix}$$

Letting

$$\Sigma_I \equiv \text{var}(P_s, X, \epsilon_i) = \begin{pmatrix} \sigma_P^2 & mb\sigma_x^2 & b\sigma_\epsilon^2 \\ mb\sigma_x^2 & \sigma_x^2 & 0 \\ b\sigma_\epsilon^2 & 0 & \sigma_\epsilon^2 \end{pmatrix}$$

we have that

$$\mathbb{E}[-\exp(-\tau W_I)] = -|I + \tau \Sigma_I A_I|^{-1/2}.$$

Including the cost $c$, the ex-ante expected utility for the informed agent is $-|I + \tau \Sigma_I A_I|^{-1/2} \exp(\tau c)$. The equilibrium number of informed traders, ignoring integer problems, is therefore given by the condition

$$-|I + \tau \Sigma_U A_U|^{-1/2} = -|I + \tau \Sigma_I A_I|^{-1/2} \exp(\tau c),$$

which reduces to the expression in the Lemma.

**Proof of Proposition 1.**

First we show sufficiency. If $\alpha = \beta$, then from (15) in Lemma 14 we see that $\lim_{N \to \infty} r = \frac{1}{\tau \sigma_\epsilon^2} > 0$. From (6) it is immediate that the limiting prices are partially revealing and have the form given in the Proposition as long as $\beta > 0$. We show necessity by contradiction. First suppose that $\alpha < \beta$. Then (15) yields $\lim_{N \to \infty} r = \frac{1}{\tau \sigma_\epsilon^2} > 0$. From (6) it is immediate that the limiting prices are completely uninformative about $X$. Now suppose that $\alpha > \beta$. If $\beta \geq \alpha/2$ we again have that $\lim_{N \to \infty} r > 0$, and by inspection of (6) we see that prices become fully revealing as $N \to \infty$. If $\beta < \alpha/2$, first note that, letting $\gamma = \beta - \alpha/2 < 0$, from (18) we have

$$r = \left(\frac{N \gamma \Sigma^2}{\tau \mu \sigma_\epsilon^2}\right)^{1/3} k(N),$$

where $\lim_{N \to \infty} k(N) = 1$, or $r \approx N^{\gamma/3} c$, for some constant $c > 0$. The coefficient that multiplies $X$ in (6) is therefore approximately of the order $N^{(\alpha - \beta/2)/3}$, whereas that on $\hat{e}$ is of the order $N^{(\alpha/2 - \beta)/2}$. Some simple calculations show that as long as $\alpha > 0$ the dominant term is the one on $X$, so that prices become fully revealing. This shows that $\alpha = \beta$ is a necessary condition for prices to be asymptotically partially revealing.

**Proof of Proposition 2.**
We first show that a necessary and sufficient condition for \( \lim_{N \to \infty} \lambda_\alpha(N) = \lambda_\alpha^* > 0 \) is that \( \alpha = \beta \). In particular, we show that unless \( \alpha = \beta \) there does not exist such \( \lambda_\alpha^* \) that meets the limiting equilibrium condition:

\[
\lim_{N \to \infty} \frac{|I + \tau \Sigma I(m^*(N))A_I(m^*(N))|}{|I + \tau \Sigma U(m^*(N))A_U(m^*(N))|} = e^{2\tau c}.
\] (20)

Consider first the case \( \alpha > \beta \). We first assume the existence of a limiting information acquisition equilibrium, and show by contradiction that such equilibrium cannot exist. As shown in Proposition 1 prices become fully revealing in this case. Therefore the expected utility of the informed will be strictly lower than the expected utility of the uninformed by the costs of information gathering (a factor \( e^{\tau c} \)). In terms of (20) we have

\[
\lim_{N \to \infty} \frac{|I + \tau \Sigma I(m^*(N))A_I(m^*(N))|}{|I + \tau \Sigma U(m^*(N))A_U(m^*(N))|} = 1;
\]

by formally taking limits in the expressions from Lemma 2. Therefore \( \alpha > \beta \) cannot be an equilibrium.

In the case of \( \alpha < \beta \), prices in the limit are completely uninformative about \( X \), as shown in the proof of Proposition 1. The following are the limiting expressions for equilibrium price coefficients:

\[
\begin{align*}
    r &\to \frac{1}{\tau \sigma_e^2}; \\
    d &\to \frac{\lambda_\alpha r \sigma_x^2 N^{\alpha/2-\beta}}{\Sigma_Z^2}; \\
    b &\to rd; \\
    w &\to 0; \\
    q &\to -r; \\
    \sigma_p^2 &\to 0.
\end{align*}
\]

Prices converge to zero, yet there exists a subset of traders that trade with the noise traders and profit from them: note that the limiting trading strategies for the informed are non-trivial.

The above expressions immediately imply that

\[
\lim_{N \to \infty} |I + \tau \Sigma U A_U| = 1;
\]

\[
\lim_{N \to \infty} |I + \tau \Sigma I A_I| = 1 + \frac{\sigma_x^2}{\sigma_e^2};
\]

and therefore (20) cannot be met when \( \mathcal{E}(\beta) \) has a non-trivial information acquisition technology.

Now we proceed to show that for \( \alpha = \beta \) and \( \beta \in (0,2) \) there exists such a limiting equilib-
rium, and characterize it explicitly. First note that taking limits in (14) we have

\[
\lim_{N \to \infty} N^{\beta/2} d = \hat{d}_\beta \frac{\lambda_\beta r \sigma_x^2}{[(\lambda_\alpha r)^2 \sigma_x^2 + \Sigma^2_Z]^2};
\]

\[
\lim_{N \to \infty} \sigma_P^2 = \hat{d}_\beta^2 (\lambda_\alpha r)^2 \sigma_x^2 + \Sigma^2_Z; \]

\[
\lim_{N \to \infty} r = \frac{1}{\tau \sigma_x^2};
\]

\[
\lim_{N \to \infty} b = 0; \]

\[
\lim_{N \to \infty} bm = \lambda_\beta r \hat{d}_\beta; \]

\[
\lim_{N \to \infty} w = 0; \]

\[
\lim_{N \to \infty} q = -\frac{1}{\tau \sigma_x^2}.
\]

Using the notation from Lemma 2 we have that

\[
\lim_{N \to \infty} A_I = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix};
\]

\[
\lim_{N \to \infty} \Sigma_I = \begin{bmatrix} \sigma_P^2 & \hat{b}\sigma_X^2 & 0 \\ \hat{b}\sigma_X^2 & \sigma_X^2 & 0 \\ 0 & 0 & \sigma_\epsilon^2 \end{bmatrix};
\]

\[
\lim_{N \to \infty} I + \tau \Sigma_I A_I = \begin{bmatrix} 1 + 2\tau r (\sigma_P^2 - \hat{b}\sigma_X^2) & -2\tau r (\sigma_P^2 - \hat{b}\sigma_X^2) & -\tau r (\sigma_P^2 - \hat{b}\sigma_X^2) \\ -2\tau r \sigma_X^2 (1 - \hat{b}) & 1 + 2\tau r \sigma_X^2 (1 - \hat{b}) & \tau r \sigma_X^2 (1 - \hat{b}) \\ -1 & 1 & 1 \end{bmatrix}
\]

Some fairly straightforward calculations yield

\[
\lim_{N \to \infty} |I + \tau \Sigma_I A_I| = 1 + \tau r [\sigma_P^2 + \sigma_X^2 - 2\hat{b}\sigma_X^2]
\]

\[
= 1 + \frac{1}{\sigma_\epsilon^2} \left[ \hat{d}_\beta^2 \Sigma^2_Z + (1 - \hat{b})^2 \sigma_X^2 \right]
\]

Further note that

\[
1 - \hat{b} = \frac{1/\sigma_X^2}{(1/\sigma_X^2 + \lambda_\alpha r^2 / \Sigma^2_Z)}
\]
\[ d^2 \Sigma_Z^2 = \frac{\lambda^2 \tau^2 / \Sigma_Z^2}{(1/\sigma_X^2 + \lambda^2 \tau^2 / \Sigma_Z^2)} \]

so that

\[ \lim_{N \to \infty} |I + \tau \Sigma I A| = 1 + \frac{1/\sigma^2}{\frac{1}{\sigma^2} + \lambda^2 \tau^2 \Sigma_Z^2}. \]

Moreover, using the previous limiting expressions and Lemma 2

\[ \lim_{N \to \infty} |I + \tau \Sigma U A| = 1. \]

The equilibrium amount of informed agents is given by

\[ \lim_{N \to \infty} \frac{|I + \tau \Sigma I A|}{|I + \tau \Sigma U A|} = e^{2\tau \epsilon}; \]

so we are left with showing that there exists \( \lambda_\beta \) such that

\[ \frac{1/\sigma^2}{\frac{1}{\sigma^2} + \lambda^2 \tau^2 \Sigma_Z^2} = C. \]

Some straightforward calculations show that if the economy has a non-trivial information acquisition technology there always exists such \( \lambda_\beta > 0 \), namely

\[ \lambda_\beta = \frac{\Sigma_Z}{r} \sqrt{\frac{1}{\sigma^2 C} - \frac{1}{\sigma^2 X}} \]

which concludes the proof for the case \( 0 < \beta < 2 \).

The case \( \beta = 2 \) follows from Verrecchia (1982) (see also Diamond (1985)). □

**Proof of Proposition 3.**

Some tedious calculations show that the ratio of the determinants in equation (19) is independent of \( n \), the number of uninformed agents. Therefore the equilibrium number of traders \( m^* \) in Lemma 2 is independent of \( n \). The Proposition immediately follows from this observation. □

**Proof of Proposition 4.**
Take an economy with an arbitrary number of agents $N$, and consider the information acquisition decision of two agents, one who is informed with risk-aversion parameter $\tau_I$, and a second who is uninformed with risk-aversion parameter $\tau_U$. We will argue that in the limit $\tau_U \geq \tau_I$ by contradiction. Consider an equilibrium in which agents with $\tau_I$ become informed, but agents of risk-aversion $\tau_U$ don’t, and $\tau_I > \tau_U$. We next show that this set of strategies cannot be an equilibrium for $N$ large enough. Note that in any equilibrium it must be the case that

$$-|I + \tau_I \Sigma I A I|^{-1/2}e^{\tau_I c} \geq -|I + \tau_I \Sigma U A U|^{-1/2};$$  \hspace{1cm} (21)

$$-|I + \tau_U \Sigma U A U|^{-1/2} \geq -|I + \tau_U \Sigma I A I|^{-1/2}e^{\tau_U c}. \hspace{1cm} (22)$$

Equation (21) simply states that the informed agent is better off by buying the signal than by not purchasing it, and (22) requires that the uninformed agent will not desire to buy a signal.

Taking limits, from Proposition 2 we have that the above inequalities become:

$$-\left(1 + \frac{1}{\sigma^2} + \frac{\lambda^2 r^2}{\Sigma^2 Z} \right)^{-1/2} e^{\tau_I c} \geq -1$$ \hspace{1cm} (23)

$$-1 \geq -\left(1 + \frac{1}{\sigma^2} + \frac{\lambda^2 r^2}{\Sigma^2 Z} \right)^{-1/2} e^{\tau_U c} \hspace{1cm} (24)$$

or $e^{\tau_I c} \leq e^{\tau_U c}$, i.e. $\tau_I < \tau_U$, which is a contradiction.

The above argument shows that any set of strategy profiles in which an agent $\tau_{min}$ does not acquire information, but an agent with risk-aversion $\tau_I > \tau_{min}$ does, can not be an equilibrium for $N$ large enough. The rest of the proof follows the same lines as Proposition 2. □

**Proof of Proposition 5.**

The proof is very similar to that of Proposition 2, so we simply sketch the necessary steps. First, we write out the equilibrium conditions fixing $m$ and $N$. Letting $m = \alpha N^{\alpha/2}$ we explicitly compute the limiting equilibria for $\alpha = 1$ and $\alpha \neq 1$. The same arguments as before yield that the only equilibria is that with $\alpha = 1$. For simplicity in the exposition we assume $\tau = 1$ in what follows (the extension to general $\tau$ is immediate).
In order to characterize the equilibrium values for the price coefficients \((b, d_I, d_U)\) in (13), fixing \(m\) and \(N\), first note that by the usual projection theorem we can always write the optimal trading strategy for each type of agent as

\[
\theta_i = \begin{cases} 
 rY_i + qP_s + tZ_i; & i = 1, \ldots, m; \\
 wP_s + vZ_i; & i = m + 1, \ldots, N.
\end{cases}
\]

Carrying out the calculations we have that

\[
\begin{align*}
 r &= \frac{\Sigma_Z^2(d_I^2(m - 1) + d_U^2(N - m))}{\xi_I} \\
 t &= \frac{bd_I\sigma_c^2(m - 1)}{\xi_I} \\
 q &= \frac{b(m - 1)\sigma_c^2}{\xi_I} - \frac{\kappa_I}{\sigma_x^2\xi_I} \\
 v &= \frac{bmd_U\Sigma_z^2}{\xi_U} \\
 w &= \frac{bm\Sigma_z^2}{\xi_U} - \frac{\kappa_U}{\xi_U\sigma_x^2}
\end{align*}
\]

and

\[
\begin{align*}
\kappa_I &= \Sigma_Z^2(\sigma_x^2 + \sigma_c^2)(d_I^2(m - 1) + d_U^2(N - m)) + \sigma_x^2\sigma_c^2b^2m(m - 1) + b^2\sigma_c^4(m - 1) \\
\kappa_U &= \Sigma_Z^2(m^2b^2\sigma_x^2 + mb^2\sigma_c^2 + \Sigma_Z^2(d_I^2m + d_U^2(N - m - 1))) \\
\xi_I &= \sigma_c^2[b^2(m - 1)\sigma_c^2 + \Sigma_Z^2((m - 1)d_I^2 + (N - m)d_U^2)] \\
\xi_U &= \Sigma_Z^2[mb^2\sigma_c^2 + \Sigma_Z^2(d_I^2m + d_U^2(N - m - 1)]
\end{align*}
\]

Market clearing requires

\[
w(N - m)P_s + v \sum_{i=m+1}^{N} Z_i + r \sum_{i=1}^{m} Y_i + qmP_s + t \sum_{i=1}^{m} Z_i = \sum_{i=1}^{N} Z_i
\]

or, letting \(\Lambda = -[qm + w(N - m)]\)

\[
r \sum_{i=1}^{m} Y_i + (v - 1) \sum_{i=m+1}^{N} Z_i + (t - 1) \sum_{i=1}^{m} Z_i = \Lambda P_s = \Lambda \left( \sum_{i=1}^{m} b_iY_i - d_I \sum_{i=1}^{m} Z_i - d_U \sum_{i=m+1}^{N} Z_i \right)
\]
Matching coefficients one gets that the equilibrium values for \((b, d_U, d_I)\) are given by the solution to

\[
\begin{align*}
    b \Lambda &= r; \quad (25) \\
    d_I \Lambda &= 1 - t; \quad (26) \\
    d_U \Lambda &= 1 - v. \quad (27)
\end{align*}
\]

It does not seem possible to solve the system of non-linear equations (25)-(27) in closed-form, but fortunately it is still possible to characterize the solution in the limiting case. In particular, it is straightforward (but tedious) to check that if \(m^* = \lambda^E_1 N^{1/2}\) for some \(\lambda^E_1 > 0\), then the price coefficients converge to those in Proposition 2. In particular we have that

\[
\begin{align*}
    \lim_{N \to \infty} b &= \lim_{N \to \infty} d_U = \frac{1}{\tau \sigma^2} \\
    \lim_{N \to \infty} d_I N^{1/2} &= \lim_{N \to \infty} d_U N^{1/2} = d^*_1.
\end{align*}
\]

Moreover, by equating the expected utilities of the uninformed and informed we again arrive at the conclusion that \(\lambda^E_1 = \lambda^*_1\) in equilibrium.

As in the proof of Proposition 2, it is straightforward to check that allowing for \(m^* = \lambda_\alpha N^{\alpha/2}\) does not increase the set of equilibria: only for \(\alpha = 1\) do we have a Nash equilibrium at the information acquisition stage for \(N\) large enough. \(\square\)
References


Equilibrium with information acquisition \( \lambda^*_\beta \)

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<tr>
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<th>( \beta = 0 )</th>
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Table 1: Comparison of limiting equilibria as a function of \( \beta \).