Abstract

We consider a general mechanism design setting where each agent can acquire (covert) information before participating in the mechanism. The central question is whether a mechanism exists which provides the efficient incentives for information acquisition ex ante and implements the efficient allocation conditional on the private information ex post.

It is shown that in every private value environment the Vickrey-Clark-Groves mechanism guarantees both ex ante as well as ex post efficiency. In contrast, with common values, ex ante and ex post efficiency cannot be reconciled in general. Sufficient conditions in terms of sub- and supermodularity are provided when (all) ex post efficient mechanisms lead to private under- or overacquisition of information.

1 Introduction

1.1 Motivation

In most of the literature on mechanism design, the model assumes that a number of economic agents possess a piece of information that is relevant for the efficient allocation of resources.
The task of the mechanism designer is to find a game form that induces the agents to reveal their private information. An efficient mechanism is one where the final allocation is efficient given all the private information available in the economy.

In this paper, we take this analysis one step further. We assume that before participating in the mechanism each agent can covertly obtain additional private information at a cost. After the information has been acquired, the mechanism is executed. Hence the primitive notion in our model is an information gathering technology rather than a fixed informational type for each player. It is clear that the properties of the mechanism to be played in the second stage affect the players’ incentives to acquire information in the ex ante stage.

The main results in this paper characterize information acquisition in ex post efficient mechanisms. Efficiency of a mechanism in this paper is understood in the same sense as in the original contributions by Vickrey, Clarke and Groves. In particular, we do not impose balanced budget or individual rationality constraints on the mechanism designer. In the independent private values case, we show that the Vickrey-Clarke-Groves (henceforth VCG) mechanism induces efficient information acquisition at the ex ante stage.

The common values case is much less straightforward to analyze. In light of the recent results by Dasgupta & Maskin (2000) and Jehiel & Moldovanu (2000), it is in general impossible to find mechanisms that would induce ex post efficient allocations. Adding an ex ante stage of information acquisition does not alleviate this problem. The two basic requirements for incentive compatibility of the efficient allocation rule are that the signals to the agents be single dimensional and that the allocation rule be monotonic in the signals. Even when these two conditions are met, we show that any efficient ex-post mechanism does not result in ex ante efficient information acquisition. We use ex post equilibrium as our solution concept. An attractive feature of this concept for problems with endogenously determined information is that the mechanisms do not depend on the distributions of the signals. By the revenue equivalence theorem, any allocation rule that can be supported in an ex post equilibrium results in the same expected payoffs to all of the players as the VCG mechanism, provided that the lowest type receives the same utility in the mechanisms. But the defining characteristic of the VCG mechanism is that an agent’s payoff changes only when the allocation changes due to his announcement of the signal. As a result, the payoffs cannot reflect the direct informational effects on other agents, and hence the private and social incentives will differ in general.

We also investigate the direction in which the incentives to acquire information are distorted. We restrict our attention to the case where the efficient allocation rule can be implemented in an ex post equilibrium and derive new necessary and sufficient conditions for the ex post implementability. It turns out that under our sufficient conditions for implementability, the information acquisition problem also satisfies the conditions for the
appropriate multi-agent generalization of a monotone environment as defined in Karlin & Rubin (1956) and Lehmann (1988). As a result, we can expand the scope of our theory beyond signal structures that satisfy Blackwell’s order of informativeness to the much larger class of signals ordered according to their effectiveness as defined in Lehmann (1988). We show that in settings with conflicting interests between agent $i$ and all other agents, as expressed by their marginal utilities, every ex post efficient mechanism results in excessive information acquisition by agent $i$. With congruent interests between agent $i$ and agents $-i$, there is too little investment in information by agent $i$ at the ex ante stage.

The paper is organized as follows. The model is laid out in the next section. Section 3 presents the case of a single unit auction as an example of the general theory. The analysis of the independent private values case is given in section 4. Results on efficient ex post implementation are presented in section 5. Section 6 deals with ex ante efficiency in the common values case and section 7 concludes.

1.2 Literature

This paper is related to two strands of literature in mechanism design. It extends the ideas of efficient mechanism design pioneered by Vickrey (1961), Clarke (1971), and Groves (1973) in an environment with fixed private information to an environment with information acquisition.

Our results on ex-post efficient mechanisms in common values environments complement recent work by Dasgupta & Maskin (2000) and Jehiel & Moldovanu (2000). Dasgupta & Maskin (2000) suggest a generalization of the VCG mechanism to obtain an efficient allocation in the context of multi-unit auctions with common values. Jehiel and Moldovanu analyze the efficient design in a linear setting with multidimensional signals and interdependent allocations. We give necessary conditions as well as weaker sufficient conditions for the efficient design in a general nonlinear environment. The results here are valid for general allocation problems and not only for single or multi-unit auctions.

The existing literature on information acquisition in mechanism design is restricted almost entirely to the study of auctions. The notable exceptions are Crémé, Khalil and Rochet (1998a and 1998b) who study information acquisition in a Baron-Myerson adverse selection model and Auriol & Gary-Bobo (1999) who consider decentralized sampling in a collective decision model for a public good. For private value auction models, Hausch & Li (1991) find that first and second price auction give the same incentives to acquire information in a symmetric environment. Tan (1992) considers a procurement model where firms invest in R&D expenditure prior to the bidding stage. In the symmetric equilibrium with decreasing returns to scale, he observes again that revenue equivalence holds between first and second price auction. Stegeman (1996) shows that the second price auction induces ef-
icient information acquisition in the single unit independent private values case. Matthews (1977), (1984) consider endogenous information acquisition in a pure common values auction and analyze the convergence of the winning bid to the true value of the object when the number of bidders increases. Hausch & Li (1993) consider a common values model with endogenous entry and information acquisition. Persico (2000) compares the equilibrium incentives of the bidders to acquire information in first and second price auctions within a model of affiliated values.

2 Model

2.1 Payoffs

Consider a setting with \( I \) agents, indexed by \( i \in \mathcal{I} = \{1, ..., I\} \). The agents have to make a collective choice \( x \) from a compact set \( X \) of possible alternatives. Uncertainty is represented by a set of possible states of the world, \( \Omega = \prod_{i=1}^{I} \Omega_i \), where \( \Omega_i \) is assumed to be a finite set for every \( i \). An element \( \omega \in \Omega \) is a vector \( \omega = (\omega_i, \omega_{-i}) = (\omega_1, ..., \omega_i, ..., \omega_I) \). The prior distribution \( q(\omega) \) is common knowledge among the players. The marginal distribution over \( \omega_i \) is denoted by \( q_i(\omega_i) \) and we assume that the prior distribution \( q(\omega) \) satisfies independence across \( i \), or:

\[
q(\omega) = \prod_{i=1}^{I} q_i(\omega_i).
\]

We assume that agent \( i \)'s preferences depend on the choice \( x \), the state of the world \( \omega \), and a transfer payment \( t_i \) in a quasilinear manner:

\[
u_i(x, \omega) - t_i.
\]

We also assume that \( u_i \) is continuous for all \( i \). The mechanism designer is denoted with a subscript 0, and her utility is assumed to be:

\[
\sum_{i=1}^{I} t_i + u_0(x).
\]

The model is said to be a private value model if for all \( \omega, \omega' \) :

\[
\omega_i = \omega_i' \Rightarrow u_i(x, \omega) = u_i(x, \omega').
\] (1)

If condition (1) is violated, then the model displays common values.
2.2 Signals and Posteriors

Agent $i$ can acquire additional information by receiving a noisy signal about the true state of the world. Let $S_i$ be a compact set of possible signal realizations that agent $i$ may observe. Agent $i$ acquires information by choosing a distribution from a family of joint distributions over the space $S_i \times \Omega_i$:

$$\left\{ F^{\alpha_i}(s_i, \omega_i) \right\}_{\alpha_i \in A_i},$$  \hspace{1cm} (2)

parametrized by $\alpha_i \in A_i$. We refer to $F^{\alpha_i}(s_i, \omega_i)$ as the signal and $s_i$ as the signal realization. Since the conditional distribution of $s_i$ depends only on $\Omega_i$ and since the prior on $\Omega$ satisfies independence across $i$, $s_i$ is independent of $s_j$ for all $i \neq j$.\footnote{Since we focus on ex post equilibria, the independence assumptions for prior beliefs and signals are not necessary for the results on efficient implementation. Rather, the independence assumptions allow us to give conditions on the utility functions that lead to under and overacquisition of information in the ex ante stage.} Each $A_i$ is assumed to be a compact interval in $\mathbb{R}$. The cost of information acquisition is captured by a cost function $c_i(\alpha_i)$ and $c_i(\cdot)$ is assumed to be continuous in $\alpha_i$ for all $i$. We endow $\Delta(S_i \times \Omega_i)$ with the topology of weak convergence and assume that $F^{\alpha_i}(s_i, \omega_i)$ is continuous in $\alpha_i$ in that topology. This ensures that the marginal distributions on $S_i$ are continuous in $\alpha_i$ as well.

Agent $i$ acquires information by choosing $\alpha_i$. Each fixed $\alpha_i$ corresponds to a statistical experiment, and observing a signal realization $s_i \in S_i$ leads agent $i$ to update his prior belief on $\omega_i$ according to Bayes’ rule. The resulting posterior belief, $p_i(\omega_i | s_i)$ summarizes the information contained in the signal realization $s_i$, with:

$$p_i(\omega_i | s_i) = \frac{f^{\alpha_i}(s_i, \omega_i)}{\sum_{\omega_i' \in \Omega_i} f^{\alpha_i}(s_i, \omega_i')} ,$$

where we omit the dependence of $p_i(\omega_i | s_i)$ on the choice of $\alpha_i$. Considered as a family of distributions on $\Omega_i$ parametrized by $s_i$, we assume that $p_i(\omega_i | s_i)$ is continuous in $s_i$ in the weak topology on $\Omega_i$.\footnote{The continuity and compactness assumptions made above are sufficient to guarantee that the choice set of each agent is compact and that the objective function is continuous in the choice variable.}

A profile of signal realizations $s = (s_1, \ldots, s_I)$ leads to a posterior belief $p(\omega | s)$ which can be written by the independence of the prior belief and the signals as:

$$p(\omega | s) = \prod_{i=1}^I p_i(\omega_i | s_i).$$

In many instances, it is convenient to let the signal realization $s_i$ be directly a posterior belief $p_i(\cdot)$. The posterior belief of agent $i$ is then alternatively represented as a probability function, $p_i : \Omega_i \to [0, 1]$ with $p_i : \omega_i \mapsto p_i(\omega_i)$ or as probability vector $p_i$ of dimension $\#\Omega_i$. The experiment $\alpha_i$ can be represented directly by a joint distribution over $\omega_i$ and $p_i$. 


2.3 Efficiency

The ex-ante efficient allocation requires each individual agent $i$ to acquire the efficient amount of information and the allocation $x$ to be optimal conditional on the posterior beliefs of all agents. Since the model has quasilinear utilities, Pareto efficiency is equivalent to surplus maximization. The social utility is defined by

$$u(x, \omega) \triangleq \sum_{i=0}^{I} u_i(x, \omega).$$

The expected social surplus of an allocation $x$ conditional on the posterior belief $p(\omega)$ is given by:

$$u(x, p) \triangleq \sum_{\omega \in \Omega} u(x, \omega) p(\omega). \quad (3)$$

The ex-post efficient allocation $x(p)$ maximizes $u(x, p)$ for a given $p$. Given the assumptions made in the previous subsection, it is clear that a maximizer exists for all $p$.

Similarly, denote by $p_{-i}$ the information held by all agents but $i$, with $p_{-i}(\omega) = q_i(\omega_i) \prod_{j \neq i} p_j(\omega_j)$ and let $x_{-i}(p_{-i})$ be the allocation that maximizes the expected social value $u_{-i}(x, p_{-i})$ of all agents excluding $i$, with:

$$u_{-i}(x, \omega) \triangleq \sum_{j \neq i} u_j(x, \omega), \quad (4)$$

and

$$u_{-i}(x, p_{-i}) \triangleq \sum_{\omega \in \Omega} u_{-i}(x, \omega) p_{-i}(\omega). \quad (5)$$

Let $F^{\alpha}(p)$ be the distribution induced on posteriors by the vector of experiments, where $\alpha = (\alpha_1, ..., \alpha_I)$ and let $c(\alpha) = \sum_i c_i(\alpha_i)$. An ex-ante efficient allocation is a vector of experiments, $\alpha^*$, and an ex-post efficient allocation $x(p)$ such that $\alpha^*$ solves

$$\max_{\alpha \in A} \int u(x(p), p) dF^\alpha(p) - c(\alpha). \quad (6)$$

Observe that since we have used the posterior probabilities as arguments in the choice rule, the optimal allocation $x(p)$ does not depend on $\alpha$. Again, given the continuity and compactness assumptions made in the previous subsection, a solution is guaranteed to exist.

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3Recall that the mechanism designer collects all the payments and receives utility from them.
3 Information acquisition in an auction

In this section, we present an example of a single unit auction with two bidders. It is meant to introduce the basic arguments for the private and common values results and to indicate how to extend the logic of the arguments to any number of agents and allocations. A similar example is discussed in Maskin (1992) with a signal space but without an underlying state space.

The set of allocations is the set of possible assignments of the object to bidders, or $X = \{x_1, x_2\}$, where $x_i$ denotes the decision to allocate the object to bidder $i \in \{1, 2\}$. The state space of agent $i$ is given by $\Omega_i = \{0, 1\}$. We begin with a private value model, where the value of the object for bidder $i$ is $u_i(x_i, \omega) = 2\omega_i$ and $u_i(x, \omega) = 0$ for $x \neq x_i$. We let the signal of agent $i$ be simply his posterior belief $p_i = \Pr(\omega_i = 1)$. The expected (ex-post) utility for agent $i$ depends on $p_i$ and $p_j$: $u_i(x_i, p_i, p_j) = 2p_i$. The direct VCG mechanism in this setting is the second price auction where bidder $i$ pays the reported valuation of bidder $j$ conditional on obtaining the object. Ex post efficiency implies that $i$ gets the object if $u_i(x_i, p_i, p_j) \geq u_j(x_j, p_i, p_j)$, i.e. if $p_i \geq p_j$. It follows that the equilibrium utility of bidder $i$, conditional on obtaining the object, is $u_i(x_i, p_i, p_j) - u_j(x_j, p_i, p_j)$. For an arbitrary fixed realization $p_j = \hat{p}$, the valuations by $i$ and $j$ are depicted in Fig. 1a as functions of $p_i$. The equilibrium net utility of bidder $i$, has the same slope in $p_i$ as the social utility as displayed in Fig. 1b.

[INSERT FIGURE 1 HERE]

Consider next information acquisition within this auction. With a binary state structure, a signal is more informative if the posteriors are more concentrated around 0 and 1. Around $\hat{p}$, a local increase in informativeness can be represented as a lottery (with equal probability) over $\hat{p} - \varepsilon$ and $\hat{p} + \varepsilon$ for some $\varepsilon > 0$. The convexity of the equilibrium net utility (see Fig. 1b) implies that information has a positive value. More importantly, the private marginal value of the lottery coincides with the social marginal value. As a result each agent acquires the socially efficient level of information. The logic of this argument extends to all private value problems as the utility $u_{-i}(x, p)$ of all agents but $i$ is constant in $p_i$.

To extend the example to a common values environment, let $u_i(x_i, \omega) = 2\omega_i + \omega_j$. The expected valuation is then $u_i(x_i, p_i, p_j) = 2p_i + p_j$ and under an efficient allocation rule $i$ gets the object when $p_i \geq p_j$. For a given $p_j = \hat{p}$, the utilities are displayed as functions of $p_i$ in Fig. 2a. The valuation of bidder $j$ now varies with $p_i$, even though it is less responsive to $p_i$.

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4The notation in this section is in minor conflict with the general notation presented in the previous section to take advantage of the binary structure of the example: (i) $p_i$ in this section is simply a scalar rather than a probability distribution and (ii) the expected gross utility is written as function of $p_i$ and $p_j$ rather than the implied probability vector $p$ over the state space $\Omega$, which is here simply: $\Omega = \{0, 1\} \times \{0, 1\}$. 

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than the valuation of \( i \). The valuations therefore satisfy a familiar single-crossing condition. However, as the valuation of bidder \( j \) varies with \( p_i \), the original VCG mechanism does not induce truth telling in equilibrium. If we were to apply the mechanism, the equilibrium utility of agent \( i \) would be \( u_i (x_i, p_i, \hat{p}) - u_j (x_j, p_i, \hat{p}) \), but for any \( p_i > \hat{p} \), there is an \( \varepsilon > 0 \) such that bidder \( i \) could lower his report to \( p_i - \varepsilon \), still get the object, and receive \( u_i (x_i, p_i, \hat{p}) - u_j (x_j, p_i - \varepsilon, \hat{p}) > u_i (x_i, p_i, \hat{p}) - u_j (x_j, p_i, \hat{p}) \). The above argument remains valid until \( p_i = \hat{p} \), where a lower report would induce an undesirable change in the allocation. Thus by asking bidder \( i \) to pay \( u_j (x_j, \hat{p}, \hat{p}) \), incentive compatibility is preserved. The equilibrium utility of agent \( i \) is then \( u_i (x_i, p_i, \hat{p}) - u_j (x_j, \hat{p}, \hat{p}) \). When we now compare the slopes of individual payoffs and social payoffs locally at \( \hat{p} \), we find that the equilibrium utility of agent \( i \) has a sharper kink than the social utility as depicted in Fig. 2b.

[Insert Figure 2 here]

As before, more information can be represented locally as a randomization over posteriors around \( \hat{p} \). In equilibrium bidder \( i \) has excessive incentives to acquire information relative to the socially optimal level as his objective function is (locally) more convex. Conversely, agent \( i \) has insufficient incentives to acquire information if \( \partial u_i (x_j, p_i, p_j) / \partial p_i < 0 \).

Next we briefly sketch how these insights generalize beyond the current example. The (single) crossing of the utilities at \( p_i = \hat{p} \) has two important implications. First, it indicates that it is socially efficient to change the assignment from agent \( j \) to agent \( i \) at \( p_i = \hat{p} \). Consequently the social utility satisfies at \( p_i = p_j = \hat{p} \):

\[
\frac{\partial u_i (x_i, p_i, p_j)}{\partial p_i} - \frac{\partial u_j (x_j, p_i, p_j)}{\partial p_i} \geq 0. \tag{7}
\]

Consider an order \( \succ \) on \( X \), such that \( x_i \succ x_j \). If the local condition (7) holds for all \( p_i, p_j \in [0, 1] \), then \( u (x, p_i, p_j) \) is supermodular in \( (x, p_i) \). The second implication of the single crossing condition is that at \( p_i = p_j = \hat{p} \):

\[
\frac{\partial u_i (x_i, p_i, p_j)}{\partial p_i} - \frac{\partial u_j (x_j, p_i, p_j)}{\partial p_i} \geq 0, \tag{8}
\]

where the latter condition is necessary for truth telling by agent \( i \). (The partial derivative of the second term is naturally equal to zero for all \( (p_i, p_j) \) in auctions without externalities). If \( u_i (x, p_i, p_j) \) is supermodular in \( (x, p_i) \), i.e. if we require (8) to hold globally for all \( p_i, p_j \in [0, 1] \), we obtain a sufficient condition for truth telling by agent \( i \). Finally, the condition for under- or overacquisition of information by bidder \( i \) was related to the responsiveness of the utility of agent \( j \), \( u_j (x_i, p_i, p_j) \) to \( p_i \). We can now restate the conditions for overacquisition of information by agent \( i \) in terms of

\[
\frac{\partial u_j (x_i, p_i, p_j)}{\partial p_i} - \frac{\partial u_j (x_j, p_i, p_j)}{\partial p_i} \leq 0, \tag{9}
\]
or equivalently that \( u_j(x, p_i, p_j) \) is submodular in \((x, p_i)\). We shall see in the subsequent sections that the supermodularity conditions for \( u(x, p_i, p_j) \) and \( u_i(x, p_i, p_j) \) are sufficient and almost necessary conditions for efficient implementation and that sub or supermodularity conditions for \( u_{-i}(x, p_i, p_{-i}) \), provide sufficient conditions for over and underacquisition of information in ex-post efficient mechanism.

4 Private Values

This section considers information acquisition in the context of independent private values. For this environment Vickrey (1961), Clarke (1971) and Groves (1973) showed in increasing generality that the ex-post efficient allocation can be implemented in a direct revelation mechanism.

**Definition 1** A direct revelation mechanism is defined by a pair \((x, t)\), where \( x \) is an outcome function, \( x : S \rightarrow X \), and \( t \) is a transfer scheme, \( t : S \rightarrow \mathbb{R}^I \).

In the private value environment we may consider without loss of generality the set of signals realizations \( S \) to be the probability simplex \( \Delta \) over the state space \( \Omega \). The efficient allocation is implemented in dominant strategies if the transfer function has the following form. For all \( i \in I \):

\[
t_i(p) = h_i(p_{-i}) - u_{-i}(x(p), p_{-i}),
\]

where \( h_i(p_{-i}) \) is an arbitrary function of \( p_{-i} \). We refer to the class of mechanisms which implement the efficient allocation with a transfer function of the form displayed in (10) as the Vickrey-Clark-Groves (VCG) mechanism.

**Definition 2** A vector of experiments, \( \alpha \), is a local social optimum if for every \( i \), \( \alpha_i \) solves

\[
\alpha_i \in \arg \max_{\alpha'_i \in A_i} \left\{ \int u(x(p), p) dF_{i}(\alpha'_i, \alpha_{-i})(p) - c(\alpha'_i, \alpha_{-i}) \right\}.
\]

Notice that local here refers to the property that \( \alpha \) solves the maximization problem for each agent separately, or Nash locality. In consequence, a local social optimum may not necessarily be a solution to the problem when the experiments of all agents are jointly maximized.

**Theorem 1 (Private Values)** With independent private values, every local social optimum can be achieved by the VCG mechanism.
Proof. See appendix. ■

With the VCG mechanism the equilibrium net utility of agent \( i \) behaves as the social utility up to \( h_i(p_{-i}) \), which does not depend on \( p_i \). It therefore follows that the decision problem of agent \( i \) with respect to the information acquisition in terms of the posterior belief \( p_i \) is equivalent to the problem faced by the social planner. An immediate consequence of Theorem 1 is

**Corollary 1** The ex-ante efficient allocation can be implemented by the VCG mechanism.

Proof. See appendix. ■

The ex-ante stage has many equilibria if there are multiple local social optima.\(^5\) It follows that the VCG mechanism uniquely implements the ex-ante efficient allocation only if there is a unique local and hence global optimum in the information acquisition stage. This efficiency result can also be generalized to environments where each agent can invest ex ante in technologies that increase their private payoffs. The ex-ante efficiency result derived for the VCG mechanism can also be extended to any ex-post efficient mechanism by the revenue equivalence theorem.

In the current model, information is acquired by all agents simultaneously. However, it is well known in statistical decision theory that a sequential decision procedure may dominate any simultaneous procedure as it economizes on the cost of information acquisition. This observation is valid in the current model as well. An important consequence of a sequential version of the VCG mechanism is that the efficient allocation is now strongly implementable as every agent acts at every node as if he were maximizing the social value function.

The essential property which allows us to prove ex-ante efficiency with independent private values is the restriction that only agent \( i \) can (efficiently) invest in information about his own utility associated with various allocations. The logical next step is therefore to ask whether efficiency can be maintained in environments where the information of agent \( i \) is relevant to the utility calculus of agent \( j \). We pursue this question in the context of the interdependent values model investigated recently by Dasgupta & Maskin (2000) and Jehiel & Moldovanu (2000). Before we analyze the information acquisition per se, we give a complete characterization of the ex-post efficient allocation and associated equilibrium utilities for each agent in the following section.

\(^5\)Tan (1992) makes a similar observation in the context of ex-ante R&D investments in procurement auctions.
5 Common Values: Ex Post Efficiency

We adopt the model of Dasgupta & Maskin (2000) to our environment with uncertainty about the true state of nature in Subsection 5.1, where we present necessary and sufficient conditions for efficient implementation with a direct revelation mechanism. Similar results are briefly stated for a continuous allocation space in Subsection 5.2.

5.1 Finite Allocation Space

We start by considering a set of finitely many allocations: \( X = \{ x^0, x^1, ..., x^N \} \). Player \( i \)'s expected utility from an allocation \( x^n \) conditional on signal realization \( s \) is, using independence, given by:

\[
u_i(x^n, s) = \sum_{\omega \in \Omega} u_i(x^n, \omega) \prod_{j \in I} p_j(\omega_j | s_j).
\]

The posterior belief \( p_i(\cdot | s_i) \) is an element of a probability simplex of dimension \( \# \Omega_i - 1 \). If we take the signal realization to be directly the posterior belief, then the signal is multidimensional and by the results of Jehiel & Moldovanu (2000), an efficient ex-post implementation does not exist in general. We therefore restrict our attention to an arbitrary class of one-dimensional signal realizations \( S_i = [s_i, \pi_i] \subset \mathbb{R} \) with the result that the associated posterior beliefs \( p_i(\cdot | s_i) \) form a one-dimensional manifold in \( \Delta(\Omega_i) \). In this section the allocation problem is analyzed exclusively at the ex-post stage. The utilities are therefore written as functions of \( (x, s) \) rather than \( (x, \omega) \) and consequently \( x(s) \) is an ex-post efficient allocation rule conditional on the signal \( s \). We assume \( u_i(x, s) \) to be continuously differentiable in \( s \) for all \( i \).

Next we present necessary and sufficient conditions for efficient implementation in an ex-post equilibrium. By the revelation principle, we can restrict ourselves to direct mechanisms and truth telling strategies.

**Definition 3** A direct revelation mechanism \((x, t)\) permits implementation in an ex-post equilibrium if \( \forall i, \forall s \in S \):

\[ u_i(x(s), s) - t_i(s) \geq u_i(x(\bar{s}_i, s_{-i}), s) - t_i(\bar{s}_i, s_{-i}), \forall \bar{s}_i \in S_i. \]

An ex-post equilibrium, while not requiring dominant strategies, remains a Bayesian equilibrium for any prior distribution over types. For the rest of this subsection, we fix the

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*Dasgupta & Maskin (2000) actually restrict attention to multi-object auctions and achieve implementation through an indirect mechanism in which the bidders report their valuations contingent on the reports by the other bidders, but not directly their signals. Jehiel & Moldovanu (2000) present sufficient conditions in a linear model for general allocation problems with a direct revelation mechanism.*
realization of the signals \( s_{-i} \) and focus on truth telling conditions for agent \( i \). Let the set \( S^n_i \) be defined as the subset of \( S_i \) for which \( x^n \) is an efficient allocation:

\[
S^n_i = \left\{ s_i \in S_i \mid u(x^n, s_i, s_{-i}) \geq u\left(x^k, s_i, s_{-i}\right), \forall x_k \neq x_n \right\}.
\]

For any two sets \( S^k_i \) and \( S^l_i \) with a non-empty intersection, we call a point \( s^{kl}_i \in S^k_i \cap S^l_i \) a \( k \text{ to } l \) change point if there exists an \( \varepsilon > 0 \) such that either:

\[
s_i \in [s^{kl}_i - \varepsilon, s^{kl}_i) \Rightarrow s_i \in S^k_i, \; s_i \notin S^l_i. \tag{11}
\]

or

\[
\forall s_i \in (s^{kl}_i, s_i + \varepsilon] \Rightarrow s_i \notin S^k_i, \; s_i \in S^l_i. \tag{12}
\]

Symmetrically, we can define \( s^{lk}_i \in S^k_i \cap S^l_i \) to be an \( l \text{ to } k \) change point. By extension, let \( s^{kl} \triangleq (s^{kl}_i, s_{-i}) \). Every change point \( s^{kl} \) has the property that at \( s = s^{kl} \):

\[
\frac{\partial u(x^k, s)}{\partial s_i} \leq \frac{\partial u(x^l, s)}{\partial s_i}. \tag{13}
\]

Consider next the ex-post truth telling condition for agent \( i \):

\[
u_i(x(s), s) - t_i(s) \geq u_i(x(\hat{s}_i, s_{-i}), (s_i, s_{-i})) - t_i(\hat{s}_i, s_{-i}), \; \forall \hat{s}_i \in S_i.
\]

It follows that the transfer payment of agent \( i \) has to be constant conditional on the allocation \( x(s) = x^n \) and we denote it by \( t^n_i \).

**Proposition 1** A necessary condition for ex-post efficient implementation is that for \( \forall k, \forall l \), at \( s = s^{kl} \):

\[
\frac{\partial u_i(x^k, s)}{\partial s_i} \leq \frac{\partial u_i(x^l, s)}{\partial s_i}. \tag{13}
\]

**Proof.** See appendix. 

The inequality (13) is a familiar local sorting condition and implies that the incentive compatible transfers are uniquely determined (up to a common constant) at the change point \( s^{kl} \) by:

\[
t^k_i - t^l_i = u_{-i}\left(x^l, s^{kl}\right) - u_{-i}\left(x^k, s^{kl}\right). \tag{14}
\]

As the transfer payments \( t^n_i \) for every allocation \( x^n \) are necessarily determined at the change points, it follows that (generically) every pair of sets \( S^k_i \) and \( S^l_i \) must have an intersection which forms a connected set as otherwise \( t^k_i \) and \( t^l_i \) would be overdetermined. The later condition can be rephrased as follows:

\[\text{The condition is written as an either/or condition as the social utility may display the same partial derivative with respect to } s \text{ for the alternatives } x^k \text{ and } x^l \text{ over an interval where the social values of the alternatives } x^k \text{ and } x^l \text{ are equal.}\]
Definition 4 The collection \( \{S^n_i\}_{n=0}^N \) satisfies monotonicity if for every \( n \):

\[
s_i, s'_i \in S^n_i \Rightarrow \lambda s_i + (1 - \lambda) s'_i \in S^n_i, \quad \forall \lambda \in [0, 1].
\]

A sufficient condition for monotonicity is that the social value \( u(x^n, s) \) be single-crossing in \( (x^n, s_i) \). If monotonicity is satisfied, then there exists an optimal policy \( x(s) \) such that \( x^n \) is chosen on a connected subset \( R^n_i \subseteq S^n_i \) and nowhere else. After possibly relabeling the indices, we can endow the allocation space \( X \) with the following order, denoted by \( \prec \):

\[
x^0 \prec x^1 \prec \ldots \prec x^N,
\]

such that for all \( s_i \in R^k_i \) and \( s'_i \in R^l_i \), with

\[
s_i < s'_i \Rightarrow x(s_i, s_{-i}) = x^k \prec x^l = x(s'_i, s_{-i}).
\]

For the remainder of this section we continue to work with the order defined by (15) and (16).

Proposition 2 A generically necessary condition for ex-post efficient implementation is monotonicity.

Proof. See appendix. \( \blacksquare \)

The class of mechanisms which implement the efficient allocation with the transfers determined by (14) is referred to as the generalized Vickrey-Clark-Groves mechanism, where we initialize \( t^0_i \) by:

\[
t^0_i \triangleq h_i (s_{-i}) - u_{-i} (x^0, (s_j, s_{-i})),
\]

for some arbitrary \( h_i (s_{-i}) \). Next we strengthen the local sorting condition to obtain sufficient conditions for ex-post implementation by extending the local to a global sorting condition.

Proposition 3 Sufficient conditions for ex post efficient implementation are:

1. monotonicity is satisfied for all \( i \) and \( s \),

---

8 The socially optimal policy \( x(s) \) is not unique as any (randomized) allocation over the set \( \{x^k, x'\} \) is optimal for all \( s_i \in S^n_i \cap S^l_i \), and in particular at the change points. Moreover for some \( x^k \) the corresponding set \( R^k_i \) may be empty and in consequence, the associated optimal allocation policy would use only a strict subset of the feasible allocations. Naturally, the order defined in (15) and (16) would then extend only over the subset of allocations selected by the allocation rule \( x(s) \).
2. for all $i$, $s$ and $n$:

$$\frac{\partial u_i(x_i^{n-1},s_i)}{\partial s_i} \leq \frac{\partial u_i(x^n,s_i)}{\partial s_i}. \quad (18)$$

**Proof.** See appendix. ■

Thus if the utility of every agent $i$ displays supermodularity in $(x^n,s_i)$ and monotonicity is satisfied, then an ex-post efficient implementation exists. We wish to emphasize that the particular order imposed on the allocation space $X$ may depend on $i$ and $s_{-i}$, and all that is required is that for every $s_{-i}$, an order on $X$ can be constructed such that the conditions above for necessity and sufficiency can be met.

It may be noted that monotonicity and supermodularity are strictly weaker than the conditions suggested by Dasgupta & Maskin (2000) in the context of a multi-unit auction. In the linear (in the signals) version of the model which is investigated by Jehiel & Moldovanu (2000) where:

$$u_i(x^n) = \sum_{j=1}^{I} u_{ij}(x^n) s_j,$$

the necessary and sufficient conditions coincide. For details we refer the reader to an earlier version of the paper (Bergemann & Välimäki (2000)).

### 5.2 Continuum of Allocations

The sorting and monotonicity conditions naturally extend to the case of a continuum of allocations. Let $X \subset \mathbb{R}$ be a compact interval of the real line. We assume that $u_i(s,x)$ is twice continuously differentiable with respect to $s_i$ and $x$. As before, fix the realization of signal $s_{-i}$ and let $x(s)$ denote the efficient allocation rule. As each $u_i(x,s)$ is twice continuously differentiable, $x(s)$ is differentiable almost everywhere. If $x(s_i,s_{-i})$ is monotonic in $s_i$, we can impose a complete order, denoted by $\prec$, on the allocation space $X$ such that the order on $X$ mirrors the order of the signal space by requiring that for all $s_i, s'_i$ and $x(s_i,s_{-i}) \neq x(s'_i,s_{-i})$:

$$s_i < s'_i \Rightarrow x(s_i,s_{-i}) \prec x(s'_i,s_{-i}). \quad (19)$$

We endow $X$ with the complete order defined by (19).

**Proposition 4** Sufficient conditions for ex-post implementation are:

1. monotonicity;
2. global sorting condition:
\[
\frac{u_i(x,s)}{s_i \partial x} \geq 0, \ \forall i, \forall s, \forall x.
\]

**Proof.** See appendix. ■

As in the discrete case, monotonicity is generically necessary. If the global sorting condition is weakened to a local sorting condition at \( x = x(s) \):
\[
\frac{u_i(x(s),s)}{s_i \partial x} \geq 0, \ \forall i, \forall s.
\]
then we obtain the corresponding necessary conditions for implementation. Proposition 4 generalizes an earlier proposition by Jehiel & Moldovanu (2000) from a linear to a nonlinear environment with one-dimensional signals. The transfer payments in the generalized VCG mechanism can be represented as:
\[
t_i(s) = -\int_{s_i}^{s_{i-1}} \frac{\partial u_i(x, s_{i-1}, s_i)}{\partial x} \frac{\partial x}{\partial v_i} dv_i + t_i(s_{i-1}, s_i), \quad (20)
\]
where \( t_i(s_{i-1}) = h_i(s_{i-1}) - u_i(x(s_{i-1}, s_i), (s_{i-1}, s_i)) \), for some arbitrary \( h_i(s_{i-1}) \).

6 Common Values: Ex-Ante Ine

In this section, we analyze the implications of ex-post efficient mechanisms for the ex-ante decisions of the agents to acquire information. This problem is addressed by extending the monotone environment for a single decision-maker defined by Karlin & Rubin (1956) to a multiple agent decision environment. The monotone environment is introduced first in Subsection 6.1 and the informational inefficiency is analyzed in Subsection 6.2.

6.1 Monotone Environment

We investigate the possibility of achieving efficient ex-ante decisions, while requiring efficiency in the second stage mechanism for an arbitrary choice of signals by the agents. This implementation requirement imposes certain restrictions on the posteriors and the utility functions. As we consider the ex-ante decision problem, the appropriate sorting and monotonicity conditions have to be formulated for the state space \( \Omega \), rather than the signal space \( S \).

The first restriction concerns the dimensionality of the posteriors induced by the class of signals \( \{ F^{\alpha_i} (s_i, \omega_i) \}_{\alpha_i \in A_i} \). We require that all signal realizations generated by signals in the family \( A_i \) lead to posterior beliefs which are on a one-dimensional manifold in the probability simplex of \( \Omega_i \). The restriction to a one-dimensional manifold shared by all signals in \( A_i \) is
necessary as the information acquisition is covert and efficient implementation is generically impossible in a multi-dimensional signal space (see Jehiel & Moldovanu (2000)). With this dimensionality restriction in place, we can assume without loss of generality that every signal realization \( s_i \) leads to a fixed posterior belief \( p_i(\omega | s_i) \) independent of the choice of signal \( \alpha_i \). (Two distinct signal choices \( \alpha_i \) and \( \alpha'_i \) differ in the frequency by which signal realizations \( s_i \) are observed.)

Next we consider the appropriate sorting and monotonicity conditions in the state space \( \Omega \). We require that for every \( i \), the allocation space \( X \) can be endowed with a complete order, denoted by \( \prec \), such that \( u_i(x, \omega_i, \omega_{-i}) \) and \( u(x, \omega_i, \omega_{-i}) \) are supermodular in \((x, \omega_i)\) for all \( \omega_{-i} \).\(^9\) Observe that the ranking of the allocations is allowed to vary with \( i \) as in the previous section, but here the ranking has to be invariant with respect to \( \omega_{-i} \). In addition we require that for all \( i \), the posterior probabilities satisfy the monotone likelihood ratio property: for all \( s'_i > s_i \) and \( \omega'_i > \omega_i \), \( p_i(\omega'_i | s'_i) p_i(\omega_i | s_i) - p_i(\omega'_i | s_i) p_i(\omega_i | s'_i) \geq 0 \). The supermodularity of \( u_i(x, \omega) \) in \((x, \omega_i)\) guarantees (globally) the sorting condition, whereas the supermodularity of \( u(x, \omega_i, \omega_{-i}) \) guarantees the monotonicity of the efficient allocation.

The monotone likelihood ratio condition implies that supermodularity in \((x, \omega_i)\) translates into supermodularity in \((x, s_i)\) when taking expectations with respect to the posterior beliefs based on the signal realizations.

**Proposition 5** Suppose the monotone likelihood ratio and supermodularity conditions hold for all \( i \), then:

1. \( u_i(x, s_i, s_{-i}) \) and \( u(x, s_i, s_{-i}) \) are supermodular in \((x, s_i)\);

2. for all \( s_i, s'_i \) with \( x(s_i, s_{-i}) \neq x(s'_i, s_{-i}) \):
   \[
   s_i < s'_i \Rightarrow x(s_i, s_{-i}) \prec x(s'_i, s_{-i})
   \]

**Proof.** See appendix. \( \blacksquare \)

### 6.2 Inefficiency

As we seek to answer whether agent \( i \) has the socially correct incentives to acquire information, we essentially compare the returns from information acquisition for the social planner and agent \( i \). The information provided by a signal realization \( s_i \) affects both the valuation of any particular allocation \( x \) as well as the choice of the (socially) optimal allocation \( x(s) \). By

\(^9\)In the monotone environment of Karlin & Rubin (1956), the utility function of the decision maker was only assumed to be single crossing in \((x, \omega_i)\). The stronger condition of supermodularity is imposed here as we consider a multi-dimensional signal space and when taking the expectations over \( \omega_{-i} \), supermodularity in \((x, \omega_i)\) is preserved while the single-crossing property is not.
incentive compatibility, the generalized VCG mechanism guarantees that the social utility and the private utility of agent \( i \) are evaluated at \( x(s) \) for every \( s \). In the private values environment, the congruence between social and private utilities went even further since as functions of \( s \), they were identical up to a constant, possibly dependent on \( s_{-i} \).

In the common value environment, in contrast, the marginal utility of \( s_i \) is in general different for the social utility and the private utility of agent \( i \). This discrepancy is due to the fact that with common values, the utility of all agents but \( i \), \( u_{-i}(x,s) \), is responsive to the signal realization \( s_i \). As a result, the social planner’s preferences for information are in general different from the individual preferences. In order to determine how this discrepancy affects the incentives to acquire information, we make use of the characterization of the transfer function associated with the generalized VCG mechanism.

**Theorem 2 (Inefficiency in Mechanisms)**

*Every ex-post efficient mechanism*

1. leads (weakly) to underacquisition of information by agent \( i \) if \( u_{-i}(x,\omega_i,\omega_{-i}) \) is supermodular in \( (x,\omega_i) \).

2. leads (weakly) to overacquisition of information by agent \( i \) if \( u_{-i}(x,\omega_i,\omega_{-i}) \) is submodular in \( (x,\omega_i) \).

**Proof.** See appendix.

Here we give a brief outline of the proof. The key element in the proof is the difference between the social utility function and agent \( i \)’s private utility function. We show that this difference \((i)\) has a global maximum at \( x(s) \) and \((ii)\) is supermodular in \( (x,s_i) \). The first attribute holds locally in the generalized VCG mechanism and can be suitably extended to a global property. In consequence we can view the difference as an objective function of a decision-maker whose optimal allocation policy coincides with \( x(s) \). The difference between social and private utility is composed of the gross utility of all agents but \( i \) and the transfer payment of agent \( i \). The latter is constant in \( s_i \) conditional on \( x \) in the generalized VCG mechanism and hence the second attribute follows by the hypothesis of supermodularity of \( u_{-i}(x,\omega) \) in \( (x,\omega_i) \) after using Proposition 5. Finally, since the difference satisfies the same supermodularity conditions as \( u(x,\omega) \) and \( u_i(x,\omega) \), we can order signal structures in their informativeness according to the criterion of effectiveness suggested by Lehmann (1988). As the difference is increasing in the effectiveness order, we know that the marginal value of information is larger to the social utility than to agent \( i \)'s private utility.

The inefficiency results in Theorem 2 are stated simply in terms of the marginal utility of the remaining agents after excluding agent \( i \). If the marginal preferences of the complement set to \( i \) are congruent (in their direction) with agent \( i \), then \( i \) has insufficient incentives to
acquire information. With congruent marginal preferences, the ex-post efficient mechanism induces positive informational externalities which lead agent $i$ to underinvest in information. If the marginal utilities of agent $i$ and all the remaining agents move in opposite directions, then the resulting negative informational externality leads the agent to overinvest in information.

Finally, we wish to emphasize that the inefficiency results are again only local results in the sense that we compare the decision of agent $i$ with the planner’s decision for agent $i$, when both take the decisions of the remaining agents as given. In particular, the theorem is not a statement about the (Nash) equilibrium decisions of the agents. The interaction between the information structures chosen by the agents may conceivably lead to an equilibrium outcome in which all agents acquire too much information relative to the social optimum even though the local prediction based on $u_{-i}(x, \omega)$ being, say, supermodular in $(x, \omega_i)$ for all $i$ is that all agents acquire too little information. In this case, the theorem would still tell us that relative to the equilibrium information structure $\alpha_{-i}$, the social planner would like agent $i$ to acquire more information than $i$ chooses to in equilibrium.

We conclude this section with a generalization of the single unit auction model presented earlier to a finite number of bidders and a finite state space. In a single unit auction, the feasible allocations are simply the assignments of the object to the various bidders, and we denote by $x_j$ the assignment of the object to agent $j$. The nature of the inefficiency in the information acquisition can then be decided on the basis of the properties of the utility function of each bidder at $x_j$: $u_j(x_j, \omega)$. The utility of agent $j$ is trivially zero for $u_j(x_k, \omega)$ for all $x_k \neq x_j$.

**Theorem 3 (Inefficiency in Auctions)**

*Every ex-post efficient single-unit auction*

1. leads (weakly) to underacquisition of information by agent $i$ if $u_j(x_j, \omega_i, \omega_{-i})$ is non-increasing in $\omega_i$ for all $j \neq i$.

2. leads (weakly) to overacquisition of information by agent $i$ if $u_j(x_j, \omega_i, \omega_{-i})$ is non-decreasing in $\omega_i$ for all $j \neq i$.

**Proof.** See appendix. ■

**7 Conclusion**

This paper considers the efficiency of information acquisition in a mechanism design context. In the private values world, any mechanism which implements the efficient allocation,
also leads to an efficient level of information acquisition by the agents ex-ante. The efficiency results with private values also extend to a setting where the information is acquired sequentially before a final social allocation is implemented.

The common value model we investigated here is one where the components \( \omega_i \) of the state of the world \( \omega = (\omega_1, ..., \omega_I) \) are distributed independently. The results in this paper can be generalized to settings including ones where the prior distribution \( q(\omega) \) displays correlation across \( \Omega_i \) and the signals are one-dimensional and independent conditional on the state of the world \( \omega \), provided we make the appropriate assumptions on utilities in terms of allocations and signals directly to guarantee ex-post implementation. If we move away from ex-post implementation to Bayesian implementation, the mechanisms suggested by Cremer and McLean (1985, 1988) can be adapted to our environment to induce efficient information acquisition in models with correlated signals.

Finally, this paper considered information acquisition with a fixed number of agents. It may be of interest to investigate the limiting model as the number of agents gets large. Intuitively, one might expect that the problem of each individual agent might be closer to the private value model. If the responsiveness of the marginal utility of all other agents to the signal of agent \( i \) declines, then the sub- or supermodularity of \( u_{-i}(x,s) \) in \((x,s_i)\) may vanish and yield efficiency in the limit.
8 Appendix

The appendix collects the proofs to the propositions and theorems in the main body of the text.

**Proof of Theorem 1.** A necessary and sufficient condition for a local social optimum \( \alpha \) is that for all \( i \), \( \alpha_i \) solves

\[
\alpha_i \in \arg \max_{\alpha'_i \in A_i} \left\{ \int u(x(p), p) dF^{(\alpha'_i, \alpha_{-i})}(p) - c(\alpha'_i, \alpha_{-i}) \right\}.
\] (21)

In contrast, the expected equilibrium utility of agent \( i \) under the VCG mechanism is maximized by \( \alpha_i \) where:

\[
\alpha_i \in \arg \max_{\alpha'_i \in A_i} \left\{ \int (u_i(x(p), p) + u_{-i}(x(p), p_{-i}) - h(p_{-i})) dF^{(\alpha'_i, \alpha_{-i})}(p) - c_i(\alpha'_i) \right\},
\]

or

\[
\alpha_i \in \arg \max_{\alpha'_i \in A_i} \left\{ \int u(x(p), p) dF^{(\alpha'_i, \alpha_{-i})}(p) - c_i(\alpha'_i) \right\},
\] (22)

where \( h(p_{-i}) \) can be omitted from the objective function using independence: \( F^{(\alpha'_i, \alpha_{-i})}(p) = F^{\alpha'_i}(p_i) F^{(\alpha_{-i})}(p_{-i}) \). The equivalence of (21) and (22) follows directly from the additive separability of the cost function \( c(\alpha) \).

**Proof of Proposition 1.** The argument is by contradiction. We suppose that condition (11) for the change point \( s_{kl} \) is met, a similar argument would apply if instead (12) would hold. Suppose that (13) doesn’t hold, then there exist some \( \epsilon > 0 \) s.t.:

\[
\begin{align*}
& u_i(x^k, s_{kl}^i - \epsilon, s_{-i}) - u_i(x^l, s_{kl}^i) < \frac{1}{2} \left[ u_i(x^k, s_{kl}^i - \epsilon, s_{-i}) - u_i(x^l, s_{kl}^i) \right].
\end{align*}
\] (23)

But at the same time we require implementation, or

\[
\begin{align*}
& u_i(x^k, s_{kl}^i - \epsilon, s_{-i}) - t_{ik}^k \geq u_i(x^l, s_{kl}^i - \epsilon, s_{-i}) - t_{il}^l,
\end{align*}
\]

and

\[
\begin{align*}
& u_i(x^k, s_{kl}^i) - t_{ik}^k \leq u_i(x^l, s_{kl}^i) - t_{il}^l,
\end{align*}
\]

which jointly imply that

\[
\begin{align*}
& u_i(x^k, s_{kl}^i - \epsilon, s_{-i}) - u_i(x^l, s_{kl}^i - \epsilon, s_{-i}) \geq u_i(x^k, s_{kl}^i) - u_i(x^l, s_{kl}^i),
\end{align*}
\]

which leads immediately to a contradiction with (23).
Proof of Proposition 2. Suppose monotonicity fails to hold. Then there exists at least one set $S^k_i$ such that for $s_i, s'_i \in S^k_i$ and for some $\lambda \in (0, 1)$, $\lambda s_i + (1 - \lambda) s'_i \notin S^k_i$, but $\lambda s_i + (1 - \lambda) s'_i \in S^l_i$. By Proposition 1, the differences $t^k_i - t^l_i$ are uniquely determined by the change points. It follows that if a set $S^k_i$ is not connected, then there are more equations (as defined by the incentive compatibility conditions at the change points) than variables, $t^m_i$’s, and generically, in the payoffs of $u_i(x,s)$, the system of equations has no solution. ■

Proof of Proposition 3. By Proposition 1, the transfers are uniquely determined up to a common constant. Consider any adjacent sets $R^{n-1}_i$ and $R^n_i$:

$$\forall s_i \in R^{n-1}_i : u_i(x^n,s) - u_i(x^{n-1},s) \leq t^n_i - t^{n-1}_i,$$

and

$$\forall s_i \in R^n_i : u_i(x^n,s) - u_i(x^{n-1},s) \geq t^n_i - t^{n-1}_i.$$

Now consider any arbitrary pair $S^k_i$ and $S^m_i$ with $x_k < x_m$. We want to show that:

$$\forall x_k \prec x_m, \forall s \in R^k_i : u_i(x^k,s) - u_i(x^m,s) \geq t^k_i - t^m_i, \tag{24}$$

as well as

$$\forall x_m \succ x_k, \forall s_i \in R^m_i : u_i(x^m,s) - u_i(x^k,s) \geq t^m_i - t^k_i.$$

Consider (24). We can expand the difference on the rhs to

$$u_i(x^k,s) - u_i(x^m,s) \geq \sum_{l=k}^{m-1} t^l_i - t^{l+1}_i. \tag{25}$$

Consider the uppermost element of the sum:

$$t^{m-1}_i - t^m_i = u_i(x^{m-1},s^m) - u_i(x^m,s^m),$$

and for all $s < s^m$,

$$t^{m-1}_i - t^m_i \leq u_i(x^{m-1},s) - u_i(x^m,s),$$

or

$$u_i(x^m,s) - t^m_i \leq u_i(x^{m-1},s) - t^{m-1}_i, \tag{26}$$

by (18). Replacing the lhs by the rhs of (26) in the inequality (25), the modified inequality becomes a priori harder to satisfy. Doing so leads to

$$u_i(x^k,s) - u_i(x^{m-1},s) \geq \sum_{l=k}^{m-2} t^l_i - t^{l+1}_i,$$
and by repeatedly using the argument in (26), (25) is eventually reduced to
\[ u_i(x_k, s) - u_i(x_k+1, s) \geq t_i^k - t_i^{k+1}, \]
which is satisfied by (18), when the transfers are as in (14).

**Proof of Proposition 4.** The sufficient conditions with a continuum of allocations can be obtained directly by considering the conditions of the discrete allocation model in the limit as the set of discrete allocations converges to the set of a continuum of allocations. The details are omitted.

**Proof of Proposition 5.** By assumption, \( u(x, \omega_i, \omega_{-i}) \) is supermodular in \((x, \omega_i)\) for every \(\omega_{-i}\). The supermodularity property is preserved under expectations:
\[ u(x, \omega_i, s_{-i}) \triangleq \sum_{\omega_{-i} \in \Omega_{-i}} u(x, \omega_i, \omega_{-i}) \prod_{j \neq i} p_j(\omega_j | s_j), \]
and a fortiori \( u(x, \omega_i, s_{-i}) \) satisfies the single crossing property in \((x, \omega_i)\). By Lemma 1 of Karlin & Rubin (1956), it follows that \( u(x, s_i, s_{-i}) \) satisfies the single crossing property in \((x, s_i)\). A similar argument applies to \( u_i(x, \omega_i, s_{-i}) \). Furthermore, by Theorem 1 of Karlin & Rubin (1956), it follows that an optimal strategy which is monotone in \( s_i \) exists. This proves the first part of the proposition.

If \( u(x, \omega_i, s_{-i}) \) is supermodular in \((x, \omega_i)\) for every \( s_{-i} \), then \( u(x, s_i, s_{-i}) \), defined as
\[ u(x, s_i, s_{-i}) \triangleq \sum_{\omega_i \in \Omega_i} u(x, \omega_i, s_{-i}) p_i(\omega_i | s_i), \]
is also supermodular in \((x, s_i)\) by Theorem 3.10.1 in Topkis (1998) since \( p_i(\omega_i | s_i) \) satisfies the monotone likelihood ratio.

**Proof of Theorem 2.** The following proof is written for a continuum of allocations, but all arguments go through with the obvious notational modifications for a finite set of allocations. We start with the net utility of agent \( i \) under the generalized VCG mechanism which is given by:
\[ u_i(x, s) - t_i(s), \]
where \( s = (s_i, s_{-i}) \) is the true signal by truth telling under the mechanism. For a fixed \( s_{-i} \), we can rewrite the transfer \( t_i(s_i, s_{-i}) \) to be determined directly by \( x \) rather than \((s_i, s_{-i})\). This is without loss of generality as we recall that \( t_i(s_i, s_{-i}) \) is constant in \( s_i \) conditional on \( x \). The net utility of agent \( i \) can now be written directly as
\[ v_i(x, s) \triangleq u_i(x, s) - t_i(x). \] (27)
The transfer function $t_i(x)$ is given by analogy with (20) as
\[
t_i(x) = - \int_x^z \frac{\partial u_i(z,s(z))}{\partial z} dz + t_i(x),
\]
where $s(x)$ is the inverse function of $x(s)$ for a fixed $s_{-i}$, or $s(z) = x^{-1}(z)$. The function $s(x)$ is well defined if $x(s)$ is strictly increasing in $s_i$. If $x$ has ‘flats’ in $s_i$, then the integral would have to be modified in the obvious way. It follows directly from (27) that $v_i(x,s)$ is supermodular in $(x,s_i)$ if and only if $u_i(x,s)$ is supermodular in $(x,s_i)$, which in turn is guaranteed by the supermodularity of $u_i(x,\omega)$ in $(x,\omega)$, as shown in Proposition 5.

Next we show that $u(x,s) - v_i(x,s)$ is (i) supermodular in $(x,s_i)$ and (ii) achieves a global maximum at $x = x(s)$ for all $s$. The first property is guaranteed by the same argument as before if $u_i(x,s)$ is supermodular in $(x,s_i)$ as
\[
u_i(x,s) = u_{-i}(x,s) + t_i(x).
\]
Observe next that $u_{-i}(x,s) + t_i(x)$ has a stationary point at $x = x(s)$ for all $s$ by (28):

\[
\frac{\partial u_{-i}(x,s)}{\partial x} + \frac{\partial t_i(x)}{\partial x} = \frac{\partial u_{-i}(x,s)}{\partial x} - \frac{\partial u_{-i}(x,s(x))}{\partial x} = 0.
\]
Notice also that locally at $x = x(s)$ the function is concave in $x$ as the second derivative with respect to $x$ is given by:

\[
\frac{\partial^2 u_{-i}(x,s)}{\partial x^2} - \frac{\partial^2 u_{-i}(x,s(x))}{\partial x^2} - \frac{\partial^2 u_{-i}(x,s(x))}{\partial x \partial s} \frac{ds(x)}{dx} \geq 0
\]
as the first two terms cancel at $s = s(x)$, and since

\[
\frac{\partial^2 u_{-i}(x,s(x))}{\partial x \partial s} \frac{ds(x)}{dx} \geq 0
\]
by the supermodularity of $u_{-i}(x,s)$ and $u(x,s)$ in $(x,s_i)$. However our standing assumptions don’t allow us to conclude that the local maximum is also a global maximum. This final obstacle can be removed by modifying the objective function $u_{-i}(x,s) + t_i(x)$ through the addition of a new function $g(x,s)$ with:

\[
G(x,s) = u_{-i}(x,s) + t_i(x) + g(x,s),
\]
such that the following properties are satisfied:

\[
g(x(s),s) = 0, \text{ for all } s; \quad (a)
\]

\[
G(x,s) \text{ is supermodular in } (x,s_i); \quad (b)
\]

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and
\[ G(x(s), s) \geq G(x, s), \forall s, x. \] (c)

If a function \( g(x, s) \) exists such that \( G(x, s) \) satisfies the properties (a) – (c), then \( G(x, s) \) satisfies assumption (i) and (ii) of Theorem 5.1 of Lehmann (1988). Moreover the expected value of \( G(x, s) \) evaluated at \( x = x(s) \) is equal to \( u_{-i}(x, s) + t_i(x) \) evaluated at \( x = x(s) \).

To accomplish this define an auxiliary function \( b(s) \) by:
\[ b(s) = u(x(s), s) - u_{-i}(x(s), s) - t_i(x(s)), \]
and define \( g(x, s) \) to be:
\[ g(x, s) = u(x, s) - u_{-i}(x, s) - t_i(x) - b(s). \]

It is now easy to verify that \( G(x, s) \) shares the supermodularity properties of \( u(x, s) \), has a global maximum at \( x = x(s) \) for every \( s \), and indeed \( g(x(s), s) = 0 \). It remains to take expectations. We maintain \( s_{-i} \) to be fixed. We take the expectation with respect to the distribution \( F_{\alpha_i} \) and \( F_{\alpha'_i} \) and denote
\[ G(\alpha_i, s_{-i}) = \mathbb{E}_{s_i} [G(x(s_i, s_{-i}), (s_i, s_{-i})) | F_{\alpha_i}] \]
and
\[ G(\alpha'_i, s_{-i}) = \mathbb{E}_{s_i} [G(x(s_i, s_{-i}), (s_i, s_{-i})) | F_{\alpha'_i}]. \]

By assumption the signal realizations \( S_i \) induce a one-dimensional manifold in the space \( \Delta_i \) of posterior beliefs independent of \( \alpha_i \in A_i \). We may therefore without loss of generality assume that every \( s_i \) induces a constant posterior belief \( p_i(\omega_i | s_i) \) across \( \alpha_i \in A_i \) and thus the decision rule \( x(s_i, s_{-i}) \) does not depend on \( \alpha_i \). It then follows by Lehmann’s theorem that if \( \alpha_i \) is more effective than \( \alpha'_i \), we have
\[ G(\alpha_i, s_{-i}) \geq G(\alpha'_i, s_{-i}). \]

From (a)-(c), we can then conclude that
\[ u_{-i}(\alpha_i, s_{-i}) + t_i(\alpha_i, s_{-i}) \geq u_{-i}(\alpha'_i, s_{-i}) + t_i(\alpha'_i, s_{-i}) \]
adopting again the notation that
\[ u_{-i}(\alpha_i, s_{-i}) + t_i(\alpha_i, s_{-i}) = \mathbb{E}_{s_i} [u_{-i}(x(s_i, s_{-i}), (s_i, s_{-i})) + t_i(x(s_i, s_{-i})) | F_{\alpha_i}], \]
and similar for \( \alpha'_i \). By (29), this is equivalent to
\[ u(\alpha_i, s_{-i}) - u(\alpha'_i, s_{-i}) \geq v_i(\alpha_i, s_{-i}) - v_i(\alpha'_i, s_{-i}). \] (30)
As the inequality holds for every \( s_{-i} \), it remains to hold after taking expectation over the realization of \( s_{-i} \), which concludes the proof. The corresponding result for submodularity can be obtained by simply reversing the inequalities.

**Proof of Theorem 3.** This theorem is a special case of Theorem 2 after introducing the following ranking for the allocations. With a single unit auction, the set of allocations is simply the assignment of the object to a particular bidder. For every \( i \), partition the set \( X \) of allocations into \( x_i \) and \( x_{-i} \) and order the assignments such that \( x_i \succ x_{-i} \). (The order among the remaining bidders is irrelevant.) By definition of the single object auction

\[
u_i(x_{-i}, \omega) = 0.\]

To verify the supermodularity property, it is therefore sufficient to examine the behavior of

\[
u_i(x_i, \omega) - \nu_i(x_{-i}, \omega),\]

as a function of \( \omega_i \). Similarly for \( u_{-i}(x, \omega) \). The result is now a direct consequence of Theorem 2.
References


