Calibrated Incentive Contracts

Sylvain Chassang∗†
Princeton University
April 10, 2011

Abstract
This paper studies a dynamic agency problem which includes limited liability, moral hazard and adverse selection. The paper develops a robust approach to dynamic contracting based on calibrating the payoffs that would have been delivered by simple benchmark contracts that are attractive but infeasible, due to limited liability constraints. The resulting dynamic contracts are detail-free and satisfy robust performance bounds independently of the underlying process for returns, which need not be i.i.d. or even ergodic.

1 Introduction
This paper considers a dynamic agency problem in which a principal hires an agent to make investment decisions on her behalf.1 The contracting environment includes limited liability, moral hazard, adverse selection, and makes very few assumptions about the underlying process for returns and information. The paper develops a robust approach to dynamic

∗Contact: chassang@princeton.edu.
†I’m indebted to Abhijit Banerjee, Roland Benabou, Faruk Gul, Stephen Morris, Wolfgang Pesendorfer, David Sraer and Satoru Takahashi for time, encouragement, and advice. The paper greatly benefited from conversations with Manuel Amador, Bruno Biais, Markus Brunnermeier, Mike Golosov, Johannes Hörner, Augustin Landier, Thomas Mariotti, Ulrich Müller, Guillaume Plantin, Esteban Rossi-Hansberg, Adam Szeidl, Juuso Toikka, Wei Xiong, Muhamet Yildiz and John Zhu. I’m grateful to seminar audiences at Berkeley, the Institute for Advanced Study, Harvard, MIT, NYU Stern, Paris School of Economics, Princeton, Toulouse School of Economics, and Sciences Po Paris for comments and suggestions. SeHyoun Ahn and Juan Ortner provided excellent research assistance.
1Throughout the paper, the principal is referred to as she, while the agent is referred to as he.
contracting whose main steps are as follows: 1) identify a simple class of high-liability static linear contracts that satisfy attractive and robust efficiency properties; 2) construct limited-liability dynamic contracts that achieve the same performance by calibrating the rewards to the agent so that they approximately satisfy key properties of the benchmark high-liability contracts. The resulting dynamic contracts—referred to as calibrated contracts—satisfy attractive performance bounds independently of the underlying process for returns. In particular, the results do not rely on any ergodicity or stationarity assumptions.

The model considers a risk-neutral principal and a risk-neutral agent. Both the principal and the agent are patient. The principal is infinitely lived, while the agent has a large but finite horizon which need not be known to the principal. In every period a fixed amount of resources is to be invested on behalf of the principal by the agent. The agent can be of two types. If the agent is talented, he can exert costly effort towards information acquisition. An untalented agent has no information beyond public knowledge. The agent privately observes his own type, the information he acquires, and his time horizon. The main constraint on contracts is limited liability: it is either impossible, or very difficult for the agent to receive negative transfers, and rewards must satisfy a pay-as-you-go constraint which rules out large deferred payments. The paper makes few assumptions on the underlying probability space, and the process for information and returns need not be i.i.d. or even ergodic: it may be that with non-vanishing probability there is a large number of periods where returns happen to be negative, or where costly information turns out to be useless.

This is a difficult environment to contract in. The principal is facing both adverse selection (the agent may or may not be talented and has durable private information over the cost-effectiveness of information acquisition) and moral hazard (the agent needs to acquire information and makes asset allocation decisions). At this level of generality, characterizing optimal contracts is unlikely to be informative and may not actually be possible if the principal has poorly specified beliefs over the environment. Instead the paper develops a robust approach to dynamic contracting which emphasizes prior-free performance bounds.
The first step of the approach identifies a suitable—although infeasible—class of benchmark contracts. This benchmark takes the form of a simple linear contract which rewards the agent a share of his externality on the principal. This contract exhibits high-liability since the agent is expected to provide compensation for the losses he causes. Regardless of the underlying process for returns and information, it satisfies the following properties: 

(i) the agent obtains positive expected rewards if and only if the principal obtains positive expected surplus; as a consequence, untalented types weakly prefer screening themselves out; (ii) the agent chooses the optimal asset allocation given his information; (iii) expected excess returns to the principal can be bounded below as a function of the maximum feasible expected excess returns; (iv) the previous properties hold for continuation play from the perspective of any history. In addition, a converse holds: benchmark contracts are the only contracts that satisfy these properties exactly.

The second step of the approach is to develop a simple class of dynamic contracts that robustly approximate the performance of linear high-liability contracts while satisfying severe limited liability constraints. The key insight is to calibrate both the rewards to the agent and the share of total wealth he is investing, so that for all possible strategies and all realizations of uncertainty, the payoffs obtained by the agent and the excess returns obtained by the principal remain as tightly linked as they are under benchmark linear contracts.

Taking the agent as given, these calibrated contracts induce performance approximately equal to that achieved by the benchmark linear contracts. A penalized version of the same contracts, achieves screening at a moderate performance loss. Under these penalized contracts, the agent is charged a small initial participation fee and only obtains rewards in periods where his performance is above a hurdle which depends on the magnitude of his trading activity. Agents who trade often for moderate returns find it difficult to pass this hurdle, whereas agents that can deliver moderate returns while trading rarely are hardly impeded.

The calibrated contracts described in this paper share many features with the high-
watermark contracts encountered in the hedge-fund industry. Under both high-watermark contracts and the contracts developed in this paper, the agent only gets rewarded when the current overall surplus created for the principal exceeds its historical maximum. However, the contracts described here exhibit important features that high-watermark contracts do not have and which matter essentially for performance.

1. The share of total wealth managed by the agent is calibrated as a function of both his performance and the rewards he has obtained. This needed to keep tight the relation between payoffs to the agent and payoffs to the principal.

2. The returns generated by the agent are benchmarked using the counterfactual returns the principal would have obtained on her own. This allows for screening to hold under very general conditions.

3. The agent is only rewarded when his performance exceeds a hurdle which increases with the magnitude of positions he has been taking. This favors agents delivering good performance while taking relatively small positions, and allows the screening of untalented agents at a moderate incentive cost to talented agents.

Inversely, a surprising aspect of the contracts developed in this paper is that they do not exhibit some of the features typically deemed necessary to emulate the performance of high-liability contracts: the agent is not required to hold the underlying asset allocation and there are no clawback provisions. Still, while the paper shows that these features are not necessary for good performance when the agent’s horizon is long, it would be wrong to infer that they are not useful, for instance when the agent’s horizon is short.

The paper hopes to usefully complement the rich literature on optimal dynamic contracting (see for instance Rogerson (1985), Holmström and Milgrom (1987), Spear and Srivastava (1987), Laffont and Tirole (1988), and more recently Battaglini (2005), DeMarzo and Sannikov (2006), Biais et al. (2007, 2010), DeMarzo and Fishman (2007), Sannikov (2008),
Edmans et al. (2010) or Zhu (2010)). Because optimal contracts depend finely on the details of the underlying environment, this literature has delivered rich positive predictions on how contract form should vary with the circumstances. However, a limitation of the optimal contracting approach is that it provides little guidance on how well those contracts perform if the environment is misspecified. The current paper gives up on optimality and develops a class of detail-free contracts that satisfy attractive efficiency properties for a very broad class of stochastic environments. Notably, the performance bounds satisfied by these robust contracts hold in environments where solving for optimal contracts has proved particularly difficult. This includes non-stationary environments (as in Battaglini (2005), Tchistyi (2006), He (2009), Pavan et al. (2010) or Garrett and Pavan (2010)), and settings with both moral hazard and adverse selection (as Sannikov (2007) or Fong (2008)). Still, the contracts developed in the current paper are substantially connected to the optimal contracts derived by DeMarzo and Sannikov (2006) or Biais et al. (2007, 2010) in specific settings. The similarities as well as the differences are instructive.

Because this paper focuses on the case of patient players with a long horizon, it fits in a class of Folk Theorem results in mechanism design settings. It is most closely related to Radner (1981, 1985) which proves the existence of approximately first-best contracts in a dynamic moral hazard problem where the agent’s production function is stationary and common knowledge. More recently Jackson and Sonnenschein (2007) propose simple quota mechanisms that approximately implement any Pareto efficient allocation rule in a class of dynamic multi-agent allocation problems where the agents’ preferences are i.i.d. Escobar and Toikka (2009) provide an extension to the case where preferences follow an irreducible Markov chain. As in these previous approaches, the main idea of the current paper is to constrain payoffs to satisfy key properties that would hold under an ideal benchmark. The central difference between the current paper and Radner (1981, 1985), Jackson and Sonnenschein (2007) or Escobar and Toikka (2009), is that they assume the states of the world follow an ergodic process and their analyses rely strongly on this assumption: the basic idea is
to make sure that the empirical distribution of realized outcomes matches the anticipated distribution of outcomes under first-best behavior. This approach is not applicable in the current paper since the underlying environment need not be ergodic and no law of large numbers need apply.

The methods used in the paper, as well as the emphasis on general stochastic processes, connect the paper to the literature on testing experts (see for instance Foster and Vohra (1998), Fudenberg and Levine (1999), Lehrer (2001) or more recently Al-Najjar and Weinstein (2008), Feinberg and Stewart (2008) and Olszewski and Sandroni (2008)). However, the main question here is not whether good tests are available. Rather, this paper takes a principal agent approach related to that of Echenique and Shmaya (2007), Olszewski and Peski (forthcoming) or Gradwohl and Salant (2011). These papers show that in such environments there are satisfactory ways to identify experts that generate positive surplus. Olszewski and Peski (forthcoming) relies on ex post high-liability contracts to incentivize truth telling. Gradwohl and Salant (2011) show it is possible to rely on upfront payments instead. Neither paper tackles incentive provision when information acquisition is costly, or the issue of self-screening by agents.

The paper also connects with a recent finance literature on appropriate performance measures for wealth managers. Lo (2001), Goetzmann et al. (2007) and Foster and Young (2010) all emphasize the fragility of many performance measures to gaming by the agent, as well as the difficulty of both rewarding and screening agents. In particular Foster and Young (2010) describe environments in which rewarding and screening is in fact impossible. This occurs because their environment allows for a strong form of private saving such that informed managers value income in early periods much more than in later periods. As a result, talented managers are unwilling to pay the monetary cost needed to induce screening. In contrast, the current paper essentially rules out private savings and considers patient players with constant marginal utility for income. In that case, self-screening can be obtained, even

\[\text{Specifically, consumption can be arbitrarily delayed and managers can save on their own at the same rate of returns they have for the firm.}\]
under severe limited liability constraints. The current paper is also related to recent work on the incentive properties of high-watermark contracts. Using Goetzmann et al. (2003)'s valuation of high-watermark contracts as ongoing options, Panageas and Westerfield (2009) show in a specific environment that high-watermark contracts do not necessarily lead to excessive risk taking when agents have an infinite horizon, although their payoffs are convex in returns. The current paper shows that in fact, a variation on high-watermark contracts can approximately align the interests of the agent and the principal for a large class of underlying stochastic environments.

Finally the paper is related to the literature on robust mechanism design that operationalizes the doctrine set by Wilson (1987), and attempts to characterize mechanisms that behave well under weak assumptions over payoff distributions and beliefs. A rich strand of that literature studies mechanisms that are robust with respect to the solution concept used to characterize the players' behavior. The paper is especially related to another strand in this literature, dating back to Hurwicz and Shapiro (1978) and more recently illustrated by Neeman (2003) or Hartline and Roughgarden (2008), which looks for mechanisms that satisfy robust performance bounds over broad sets of fundamentals. A tricky step, common to this literature and the current paper, is to define appropriate benchmark performance measures that allow for informative worst-case analysis of mechanisms.

The paper is structured as follows. Section 2 describes the framework. Section 3 introduces a benchmark class of high liability linear contracts that satisfy a number of attractive efficiency properties but require high levels of liability from the agent. Section 4 is the core of the paper: it develops the idea of calibrated contracts and analyzes their performance, taking the agent as given. Section 5 shows how to screen agents by means of an activity-based

---


4 Local approaches are possible and informative. For instance Madarász and Prat (2010) consider screening mechanisms that satisfy strong efficiency bounds for all type distributions within a small neighborhood. Global incentive compatibility constraints play an important role in their analysis, and will also show up in this paper.
hurdle. Section 6 relates calibrated contracts to contracts used in practice and concludes. Appendix A extends the analysis to various environments.\(^5\) Proofs are given in Appendix B, unless mentioned otherwise.

## 2 The Framework

**Players, Actions and Payoffs.** A principal (for instance, a representative investor, or a bank) hires an agent (say a wealth manager, an advisor, or a trader) to make investment allocations on her behalf. The agent is active for a large but finite number of periods \(N\). The principal has an infinite horizon and need not know the agent’s horizon \(N\). Both the principal and the agent are patient and do not discount future payoffs.\(^6\)

In each period \(t \in \{1, \cdots, N\}\), the principal invests an amount \(w\) at the beginning of the period. The amount of wealth \(w\) invested in each period is constant, and can be thought of as a steady state amount of wealth to be invested. The realized wealth \(w_t\) after investment is consumed at the end of the period, which rules out private saving. Both the principal and the agent are risk neutral over the range of flow payoffs. The agent’s outside option is set to zero.

Wealth can be invested in one of \(K\) assets whose returns at time \(t\) are denoted by \(r_t = (r_{k,t})_{k \in \{1, \cdots, K\}}\). Let \(R\) denote the set of possible returns \(r_t\). An asset allocation at time \(t\) is a vector \(a_t \in A \subset \mathbb{R}^K\) such that \(\sum_{k=1}^{K} a_t = 1\).\(^7\) Set \(A\) is convex and compact. It represents constraints on possible allocations. These constraints can be thought of as a mandate set by the principal as in He and Xiong (2010). Let \(\langle \cdot, \cdot \rangle\) denote the usual dot product. Given

---

5 Appendix A allows for varying amounts of wealth to be invested, and also considers extensions to cases where the principal is risk-averse or where the agent does not have constant marginal utility of income.

6 All results extend to the case where future payoffs are discounted with a discount factor approaching 1.

7 These assets may be financial or real assets (in the latter case, the agency problem is a project choice problem). However, because the paper imposes no intertemporal constraints on asset allocations it is most appropriate for environments where allocations are flexible and assets are liquid.
asset allocation $a_t$ and returns $r_t$, the consumer’s wealth at the end of period $t$ is

$$w_t = w \times (1 + \langle a_t, r_t \rangle).$$

For any pair of allocations $(a, a') \in A^2$, the distance between $a$ and $a'$ is defined by

$$d(a, a') \equiv \sup_{r_t \in \mathbb{R}} |\langle a - a', r_t \rangle|.$$ \hspace{1cm} (1)

The following assumption puts constraints on the set of permissible allocations $A$ and is maintained throughout the paper.

**Assumption 1.** There exists $\overline{d} \in \mathbb{R}^+$ such that for all $(a, a') \in A^2$, $d(a, a') \leq \overline{d}$.

This assumption limits the magnitude of changes that can occur with the principal getting no feedback.

There are two types of managers: talented and untalented managers. Managers know their own type. At the beginning of every period $t$, talented managers can expend cost $c_t \in [0, \overline{c}]$ towards acquiring information. This cost can be the actual cost of obtaining data, an effort cost, or the opportunity cost of time. Untalented managers only have access to public information. Managers then make an asset allocation suggestion $a_t \in A$ (whether or not they have superior information) and receive a payment $\pi_t$ depending on the realized public history at the end of period $t$. The manager’s objective is to maximize his expected average payoffs

$$\mathbb{E} \left( \frac{1}{N} \sum_{t=1}^{N} \pi_t - c_t \right).$$ \hspace{1cm} (2)

**Information.** Information acquired at time $t$ is represented as a random variable $I_t$ from a measurable state space $(\Omega, \sigma)$ to a measurable signal space $(\mathcal{I}, \sigma_{\mathcal{I}})$. Untalented managers only have access to publicly available information $I^0_t$ (which includes realized past returns). In contrast, talented managers can acquire expert information $I^e_t$ at cost $c \in [0, \overline{c}]$. The level of expenditure $c$ is a continuous choice variable, and for all $c \geq c' \geq 0$, $I^e_t$ is measurable.
with respect to $I^t_i$, i.e. the more the agent invests, the more he knows. Given an information acquisition strategy $(c_t)_{t \geq 1}$, let $(\mathcal{F}_t)_{t \geq 1}$ be the informed manager’s filtration (generated by $(I^t_i)_{t \geq 1}$), and let $(\mathcal{F}^0_t)_{t \geq 1}$ denote the public information filtration (generated by $(I^0_t)_{t \geq 1}$).

For simplicity it is convenient to assume that the principal and the agent have a common prior $P$ over the state space $(\Omega, \sigma)$.\footnote{All results hold in a non-common prior setting, taking expectations under the agent’s prior.} Let $\mathcal{P} = (\Omega, \sigma, P)$ denote the resulting probability space. The paper does not assume that either information or returns follow an i.i.d. or ergodic process. This results in a very flexible model. For instance, there may be non-vanishing probability that returns are below their period $t = 1$ expectation for an arbitrarily large number of periods. Also, the value of information that talented managers can collect may vary in arbitrary ways. For instance, once valuable trading strategies can become obsolete over time.

**Strategies.** Altogether, an agent’s strategy consists of an information acquisition strategy $c = (c_t)_{t \in \mathbb{N}}$, and an asset allocation strategy $a = (a_t)_{t \in \mathbb{N}}$, where both $c_t$ and $a_t$ are adapted to the information available to the manager at the time of decision. Let $a^0_t$ and $a^*_t$ respectively denote efficient asset allocations under information $\mathcal{F}^0_t$ and $\mathcal{F}_t$:

$$
a^0_t \in \arg \max_{a \in A} \mathbb{E}[\langle a, r_t \rangle | \mathcal{F}^0_t] \quad \text{and} \quad a^*_t \in \arg \max_{a \in A} \mathbb{E}[\langle a, r_t \rangle | \mathcal{F}_t].$$

(3)

Allocation $a^0_t$ is the allocation the principal could pick on her own, given public information $\mathcal{F}^0_t$. Let $w^0_t = w \times (1 + \langle a^0_t, r_t \rangle)$ and $w_t = w \times (1 + \langle a_t, r_t \rangle)$ denote realized wealth under allocation $a^0_t$ and under the allocation $a_t$ actually chosen by the agent.

**Limited liability contracts.** Contracts $(\pi_t)_{t \in \mathbb{N}}$ are adapted to public histories observed by the principal, where public histories consist of past public information (including past returns) as well as past suggested asset allocations by the agent. The principal has commit-
ment power but transfers are subject to the following constraints: in every period $T$,

$$\sum_{t=1}^{T} \pi_t 1_{\pi_t < 0} \geq -b, \quad \text{and} \quad \pi_T \leq \bar{\pi},$$

(4) and (5)

where $\bar{\pi}$ is a bound on rewards that is independent of $N$ but weakly greater than $w\bar{d}$, the maximum single-period gain in wealth that the agent can generate.

Condition (4) is a strong limited-liability constraint on the agent’s side. The sum of punishments that the principal can inflict on the agent is bounded above by a fixed amount $b$. Punishments may correspond to monetary transfers, as well as non-monetary costs such as grueling work, sleep deprivation, or tedious and lengthy training.

Condition (5) puts an upper bound on the per-period transfers that the principal can make (or commit to make) to the agent. This limits how long the payment of wages can be delayed and precludes the possibility of large deferred payments. Constraint (5) makes the assumption of risk neutrality more palatable, and can be viewed as a reduced form constraint excluding reward patterns that are undesirable for unmodeled reasons.\footnote{If the agent is truly risk-neutral it is optimal for the principal to delay payment until the end of the relationship: this maximizes degrees of freedom with respect to payment design. One way to motivate (5) is to consider an agent that is risk neutral over payoffs in the range $[0, \bar{\pi}]$, and has zero marginal utility of income above.}

These constraints are at the origin of the contracting problem: the agent does not share on the downside, and rewards must be given in real time rather than delayed until the end. Clearly, (4) and (5) are meant to err on the side of stringency. The surprise, in a sense, is the extent that can be achieved within these constraints.

## 3 The Benchmark: High-Liability Linear Contracts

The environment described in Section 2 involves both moral hazard and adverse selection: the agent must acquire information and makes asset allocation decisions that may or may not benefit the principal; in addition the agent’s talent and the information he may acquire
are private. At this level of generality, informative characterizations of optimal dynamic contracts are unlikely. Solving for optimal contracts may also be of limited use if the principal doesn’t have well defined beliefs over the underlying environment.

The paper embraces an alternative robust approach to dynamic contracting. The first step of the analysis defines a class of benchmark contracts that have attractive properties, but violate the limited liability constraints (4) and (5). The second step of the analysis constructs a class of dynamic contracts that do satisfy constraints (4) and (5), and also achieve performance approximately as good as that of the benchmark contracts, regardless of the underlying environment $\mathcal{P}$.

Benchmark contracts are simple linear contracts, loosely in the spirit of Vickrey-Clarke-Groves (VCG) mechanisms. Specifically, in period $t$ the agent’s reward $\pi_t$ is a share $\alpha$ of the externality his decisions have on the principal:

$$\forall t, \quad \pi_t = \alpha(w_t - w^0_t).$$

For instance, if $\alpha = .2$ and the default allocation $a^0_t$ is to invest all wealth in risk-free bonds, the benchmark contract pays the agent 20% of the excess-returns when he beats the risk-free rate, and charges him 20% of the foregone returns when he under-performs the risk-free rate. Throughout the paper, the working assumption is that the main problem with such a contract is not that it is too unsophisticated or provides insufficient incentives, but rather that it requires unrealistic levels of liability on the part of the agent.

This working assumption is motivated in two ways. First, along the lines of Holmström and Milgrom (1987) one can exhibit reasonable environments under which such linear contracts are indeed optimal. Second, and more importantly for the purposes of this paper, Theorem 1 shows that even when linear contracts are not optimal, their VCG-like features

---

11 Recall that $w_t$ and $w^0_t$ respectively denote final wealth under the agent’s suggested asset allocation and under the default, public information, asset allocation. Appendix A shows how to extend the analysis when the principal is risk-averse.
12 See also Sung (1995), Hellwig and Schmidt (2002), or Edmans and Gabaix (2009)
guarantee a number of robust efficiency properties, including a lower bound on their performance. Conversely, linear contracts are the only contracts that satisfy these properties exactly.

**An example of optimality.** Assume that in order to generate flow expected excess returns 
\[ \nu = \mathbb{E}(w_t - w_0^t | \mathcal{F}_t) \] in period \( t \), the manager must invest a cost

\[ c_t = \alpha_0 \nu \mathbf{1}_{\nu \leq p_t} + \infty \mathbf{1}_{\nu > p_t}, \]  

(7)

where \( p_t \in \mathbb{R}^+ \) can be stochastic, is unknown to the principal, and observed by the agent. There is a constant marginal cost \( \alpha_0 \) to generate expected excess returns up to an upper bound \( p_t \). Excess returns above bound \( p_t \) are infeasible. Note that because \( p_t \) can follow any stochastic process, this class of models includes environments in which talented managers can lose the ability to generate returns for extended periods of time. Clearly, any benchmark contract with reward rate \( \alpha \geq \alpha_0 \) can achieve the first best. The contract such that \( \alpha = \alpha_0 \) transfers all the surplus to the principal. Clearly, this is driven by the linearity of the production technology. Because DeMarzo and Sannikov (2006) and Biais et al. (2007, 2010) also have linear production, similar linear contracts would be optimal in their environment.

**Robust Efficiency Properties.** The main motivation for the use of linear contracts as a benchmark is that they satisfy a number of attractive efficiency properties regardless of the probability space \( \mathcal{P} \). Given a reward rate \( \alpha \), the agent solves optimization problem

\[
\max_{c,a} \mathbb{E} \left( \frac{1}{N} \sum_{t=1}^{N} \alpha(w_t - w_0^t) - c_t \right). \tag{P1}
\]
The corresponding per-period excess returns $r_\alpha$ accruing to the principal (net of payments to the agent) are

$$r_\alpha \equiv \inf \left\{ \mathbb{E}_{c,a} \left( \frac{1}{Nw} \sum_{t=1}^{N} w_t - w_0^t - \pi_t \right) \mid (c, a) \text{ solves (P1)} \right\}.$$ 

The expression for returns $r_\alpha$ involves an inf since the agent may be indifferent between multiple policy profiles. In anticipation of technical subtleties to come, it is useful to note that because the underlying environment is very general, the paper cannot rule-out binding global incentive compatibility constraints.

For any $c \in [0, \bar{c}]$, let $r_{\text{max}}(c)$ denote the maximum per-period excess returns that can be generated when the agent: 1) incurs an expected per-period cost of information acquisition $c$; 2) makes optimal asset allocation decisions given information; and 3) requires no rewards. Recalling that $a^*$ is the surplus maximizing allocation strategy given information, we have

$$r_{\text{max}}(c) \equiv \sup_{c(\cdot), s.t. \mathbb{E}[\frac{1}{N} \sum_{t=1}^{N} c_t] \leq c} \mathbb{E}_{c(\cdot), a^*} \left( \frac{1}{N} \sum_{t=1}^{N} \langle a_t^* - a_0^t, r_t \rangle \right).$$

**Theorem 1.** Regardless of probability space $\mathcal{P}$, the contract defined by (6) satisfies

(i) (no-loss): for any strategy profile $(c, a)$, expected rewards to the agent are positive if and only if expected returns to the principal are positive; as a consequence, untalented managers screen themselves out;

(ii) (conditional efficiency): given information $\mathcal{F}_t$, it is optimal for the manager to pick the efficient asset allocation $a^*$;

(iii) (lower bounds on returns): whenever the agent is rational,

$$r_\alpha \geq (1 - \alpha) \sup_{c \in [0, \bar{c}]} \left( r_{\text{max}}(c) - \frac{c}{\alpha w} \right); \quad (8)$$
(iv) (history independence): the above properties hold for continuation behavior at any history.

Point (i) ensures that regardless of the environment $\mathcal{P}$, the principal never loses surplus provided the agent gets positive expected rewards. Point (i) admits a converse: any contract that satisfies no-loss for all environments must be a linear high-liability contract.

Lemma 1. If a contract $(\pi_t)_{t \geq 1}$ is such that for all $\mathcal{P}$ and all strategies $(c, a)$ of the agent,

$$
E_{c,a} \left[ \sum_{t=1}^{N} \pi_t \right] \geq 0 \iff E_{c,a} \left[ \sum_{t=1}^{N} w_t - w_t^0 - \pi_t \right] \geq 0,
$$

then there exists $\alpha \in (0, 1)$ such that for all $t$, $\pi_t = \alpha (w_t - w_t^0)$.

Note that no-loss is a very demanding property: it requires that the agent benefits if and only if the principal benefits, for all environments and for all strategy profile $(c, a)$, including suboptimal ones. The only way to achieve this is to align the payoffs of the agent and the principal. This is related to the work of Hurwicz and Shapiro (1978) or more recently Frankel (2011) who provide max-min foundations for contracts that align the indirect preferences of the principal and the agent.

Point (ii) states that allocations are efficient conditional on information. Therefore, the only agency concern under benchmark contracts is information acquisition.

Point (iii) provides a lower bound for the returns that the principal obtains under the benchmark contract. Pick the specific value $c = \overline{c}$ corresponding to maximum effort. For any $\alpha$, as wealth under management $w$ grows arbitrarily large, the contract becomes approximately efficient, and the principal obtains a share approximately $1 - \alpha$ of the maximum returns $r_{\text{max}}(\overline{c})$. To illustrate this bound for a finite value of $w$, consider the specification $\overline{c} = $5M, $r_{\text{max}}(\overline{c}) = 5\%$, $w = $1Bn and $\alpha = 20\%$. In this case, bound (8) guarantees that the principal obtains at least 40\% of the maximum possible excess returns. Note that this

---

13This is a form of robustness to the solution concept which allows for “faulty” non-best-replying agents, as in Eliaz (2002).
is more than a share 40% of the maximum surplus since the agent must incur information acquisition costs. In addition, (8) can provide a rationale to choose $\alpha$: given a specification of mapping $c \mapsto r_{\text{max}}(c)$ (i.e. an expected production function), pick the rate $\alpha$ that maximizes the right hand side of (8). Let $\hat{\alpha}$ denote the corresponding reward rate. One can explicitly relate $r_{\hat{\alpha}}$ to first best returns $r_{FB}$.\footnote{Practically, if the surplus accruing to the principal is acceptable compared to the first best, then high-liability linear contracts of are an adequate benchmark.} Pick a first-best information acquisition level $c_{FB} \in \arg \max_{c \in [0,\pi]} wr_{\text{max}}(c) - c$. We have that $r_{FB} = r_{\text{max}}(c_{FB})$ and first-best surplus is $wr_{FB} - c_{FB}$. Maximizing the right hand side of (8) over $\alpha$ yields that the surplus $wr_{\hat{\alpha}}$ accruing to the principal under the benchmark contract of parameter $\hat{\alpha}$ satisfies

$$wr_{\hat{\alpha}} \geq (wr_{FB} - c_{FB}) \left( 1 - 2 \frac{\sqrt{c_{FB}/r_{FB}}}{\sqrt{w} + \sqrt{c_{FB}/r_{FB}}} \right).$$

Finally, point (iv) states that the attractive properties of benchmark contracts hold from the perspective of any history. Although the paper assumes full commitment, this provides reassurance that renegotiation issues are limited under benchmark contracts.

Altogether, the fact that these efficiency properties hold independently of probability space $\mathcal{P}$ motivates the use of high-liability linear contracts as a robust benchmark. The central contribution of the paper is to construct equally robust dynamic contracts that perform approximately as well as the benchmark contracts, while also satisfying limited liability constraints (4) and (5).

\section{Calibrated Contracts}

This section introduces a novel class of dynamic “calibrated” contracts that robustly approximate the performance of linear contracts while satisfying limited liability constraints (4) and (5).\footnote{Appendix A shows that static contracts with limited liability cannot approximate the performance of benchmark contracts.} This section focuses on providing a given agent with appropriate incentives.
Section 5 deals with screening. Section 6 relates calibrated contracts to contracts used in practice.

4.1 The Contract

In every period \( t \), the agent is allowed to invest a share \( \lambda_t \in (0, 1) \) of the principal’s wealth, while the remaining share \( 1 - \lambda_t \) is invested in the default asset allocation \( a_t^0 \). At the end of the period, the agent receives a payment \( \pi_t \).

Specifying investment shares and rewards \((\lambda_t, \pi_t)_{t \geq 1}\) requires additional notation. For all periods \( T \) and \( T' < T \), define

\[
\Pi_T = \sum_{t=1}^{T} \pi_t ; \quad \Sigma_T = \sum_{t=1}^{T} w_t - w_t^0 ; \quad S_T = \sum_{t=1}^{T} \lambda_t (w_t - w_t^0) \tag{9}
\]

and

\[
\Sigma_{T \setminus T'} = \sum_{t=T'}^{T} w_t - w_t^0 ; \quad S_{T \setminus T'} = \sum_{t=T'}^{T} \lambda_t (w_t - w_t^0). \tag{10}
\]

Value \( \Pi_T \) corresponds to the payoffs that the agent has obtained; \( \Sigma_T \) corresponds to the excess returns that would have been generated by fully investing according to the agent’s suggested asset allocation; \( S_T \) corresponds to the actual excess returns that have been generated given that only a share \( \lambda_t \) of wealth \( w \) is invested according to the agent’s suggestion. Values \( \Sigma_{T \setminus T'} \) and \( S_{T \setminus T'} \) compute the same quantities over time range \( \{T', \cdots, T\} \).

The difference \( \Sigma_{T \setminus T'} - S_{T \setminus T'} = \sum_{t=T'}^{T} (1 - \lambda_t) (w_t - w_t^0) \) corresponds to the foregone gains from investing only a share \( \lambda_t \) of resources according to the agent’s allocation between \( T' \) and \( T \). The difference \( \Pi_T - \alpha S_T \) corresponds to the agent’s excess rewards, the target being to reward him a share \( \alpha \) of his externality \( S_T \) on the principal. We can now define calibrated contracts formally. Using the notation \((x)^+ = \max\{0, x\}\), investment shares and rewards
$(\lambda_t, \pi_t)_{t \geq 1}$ are defined recursively as follows: $\lambda_1 = 1$, $\pi_1 = 0$, and for all $T \geq 1$

\[
\lambda_{T+1} \equiv \frac{\alpha \left[ \max_{T' \leq T} \Sigma_{T \setminus T'} - S_{T \setminus T'} \right]^+}{[\Pi_T - \alpha S_T]^+ + \alpha \left[ \max_{T' \leq T} \Sigma_{T \setminus T'} - S_{T \setminus T'} \right]^+} \equiv \frac{\alpha \times \text{maximum foregone gain}}{\text{excess rewards} + \alpha \times \text{maximum foregone gain}}
\]

with the convention that $\frac{0}{0} = 1$, and

\[
\pi_{T+1} \equiv \begin{cases} 
\alpha \lambda_{T+1}(w_{T+1} - w_0^T)^+ & \text{if } \Pi_T \leq \alpha S_T \\
0 & \text{otherwise} 
\end{cases}
\]

\[
\equiv \begin{cases} 
\alpha \lambda_{T+1}(w_{T+1} - w_0^T)^+ & \text{if } \text{rewards} \leq \alpha \times \text{actual excess returns} \\
0 & \text{otherwise} 
\end{cases}
\] (12)

Note that the contract specified above satisfies limited liability conditions (4) and (5): payments $(\pi_t)_{t \geq 1}$ are positive and bounded above by $\alpha \omega T$. Theorem 2 (stated below) shows that as horizon $N$ grows large, this class of contracts approximates the performance of benchmark contracts. Some additional notation is needed. Given a contract specification $(\lambda, \pi) = (\lambda_t, \pi_t)_{t \geq 1}$, let $r_{\lambda, \pi}$ denote the net excess returns delivered by the agent under the corresponding contract:

\[
r_{\lambda, \pi} = \inf \left\{ \mathbb{E}_{c,a} \left( \frac{1}{N} \sum_{t=1}^{N} \lambda_t (w_t - w_0^t) - \pi_t \right) \bigg| (c,a) \text{ solves } \max_{c,a} \mathbb{E} \left( \frac{1}{N} \sum_{t=1}^{N} \pi_t - c_t \right) \right\}.
\]

For any history $h_T$, net returns conditional on $h_T$ are

\[
r_{\lambda, \pi}|h_T = \inf \left\{ \mathbb{E}_{c,a} \left( \frac{1}{N} \sum_{t=T+1}^{N} \lambda_t (w_t - w_0^t) - \pi_t \bigg| h_T \right) \bigg| (c,a) \text{ solves } \max_{c,a} \mathbb{E} \left( \frac{1}{N} \sum_{t=1}^{N} \pi_t - c_t \right) \right\}. 
\]

As in Section 3, when the contract in question is the benchmark contract with parameter $\alpha$, net returns are denoted by $r_\alpha$ (similarly, let $r_\alpha|h_T$ denote the equivalent of $r_{\lambda, \pi}|h_T$ for the
benchmark contract of parameter $\alpha$).

**Theorem 2** (approximate performance). Pick $\alpha_0 \in (0, 1)$ and for any $\eta \in (0, 1)$, let $\alpha = \alpha_0 + \eta(1 - \alpha_0)$. Consider the calibrated contract $(\lambda, \pi)$ defined by (11) and (12). There exists a constant $m$ independent of time horizon $N$ and probability space $\mathcal{P}$ such that,

$$r_{\lambda, \pi} \geq (1 - \eta)r_{\alpha_0} - m\frac{1}{\sqrt{N}},$$

$$\forall h_T, \ r_{\lambda, \pi}|_{h_T} \geq (1 - \eta)r_{\alpha_0}|_{h_T} - m\frac{1}{\sqrt{N}}.$$ (14)

It follows that for $N$ large enough, the calibrated contract described by (11) and (12) generates a share approximately $1 - \eta$ of the returns the principal obtains under the benchmark contract of parameter $\alpha_0$. The mechanics underlying Theorem 2, and the reason why an additional wedge $\eta$ is needed will be discussed in Section 4.2.

Note that for Theorem 2 to hold, it is not sufficient to just reward the agent according to the payment rule $(\pi_t)_{t \geq 1}$ defined by (11) and (12). It is important that the principal actually invest shares $(\lambda_t)_{t \geq 1}$ of her wealth according to the agent’s suggestion. Indeed the reward scheme $(\pi_t)_{t \geq 1}$ does not to induce perfectly good behavior from the agent.\footnote{For instance, an agent who has lost or never had any informational advantage may pick allocations $a_t$ that are inferior to $a^0_t$, simply because they are different and, through volatility, induce a non-zero probability of reward. The investment rule $(\lambda_t)_{t \geq 1}$ insulates the principal from such misbehavior.} Rather, the payment scheme $(\pi_t)_{t \geq 1}$ reduces misbehavior to the point where it can be resolved by using the cautious investment rule specified by $(\lambda_t)_{t \geq 1}$.

### 4.2 The Mechanics of Calibrated Contracts

This idea behind calibrated contracts is to identify key incentive properties that hold under the benchmark contract and calibrate payments $(\pi_t)_{t \geq 1}$ to the agent as well as investment shares $(\lambda_t)_{t \geq 1}$ so that the same incentive properties are approximately satisfied under the calibrated contract. The properties that calibrated contracts attempt to satisfy are as follows.
For all histories $h_T$,

$$\Pi_T = \alpha S_T \quad (15)$$

$$\forall T' \leq T, \; \Sigma_{T \setminus T'} \leq S_{T \setminus T'}. \quad (16)$$

In words, the agent receives a share $\alpha$ of his actual performance $S_T$, and over any time interval \{T', \ldots, T\}, his actual performance $S_{T \setminus T'}$ (although potentially hindered by investment shares $\lambda_t \leq 1$) is at least as high as his potential performance $\Sigma_{T \setminus T'}$.\(^{17}\) Note that for any $T$, the family of inequalities (16) can be summarized by the single inequality

$$\max_{T' \leq T} \Sigma_{T \setminus T'} - S_{T \setminus T'} \leq 0.$$  

Let us now show how (11) and (12) calibrate parameters $(\lambda_t, \pi_t)_{t \geq 1}$ so that these properties hold approximately, while satisfying limited liability constraints (4) and (5). Define regrets

$$R_{1,T} \equiv \Pi_T - \alpha S_T$$

$$R_{2,T} \equiv \max_{T' \leq T} \Sigma_{T \setminus T'} - S_{T \setminus T'}.$$  

Regret $R_{1,T}$ measures how overpaid the agent has been, while regret $R_{2,T}$ measures maximum foregone profits from not fully investing according to the agent’s allocation. The goals are: 1) to keep $R_{1,T}$ small so that the agent’s reward $\Pi_T$ is a share approximately $\alpha$ of his actual externality $S_T$ on the principal; 2) to keep $R_{2,T}^+$ small, so that the foregone returns are not large.

These goals can be achieved by following the regret-minimization approach of Blackwell (1956) and Hannan (1957).\(^ {18}\) Define $R_T \equiv (R_{1,T}, \alpha R_{2,T})$ and $\rho_T \equiv R_T - R_{T-1}$ the vector

\(^{17}\)To obtain only inequality (13) in Theorem 2, it would be sufficient to consider only inequality $\Sigma_T \leq S_T$ rather than the full family of inequalities described by (16). Considering the full family of inequalities (16) yields the history-independent performance bound (14).

\(^ {18}\)See also Foster and Vohra (1999) or Cesa-Bianchi and Lugosi (2006). Regret measure $R_{2,T}$ is related to “tracking” regrets, as discussed in Cesa-Bianchi and Lugosi (2006).
of flow regrets.\(^{19}\) In order to keep regrets \(R_{1,T}\) and \(R_{2,T}\) small, it is sufficient to keep vector \(R_T\) small. This can be achieved by choosing sequences \((\pi_t)_{t \geq 1}\) and \((\lambda_t)_{t \geq 1}\) so that

\[
\forall T \geq 1, \forall w_{T+1}, \forall w^0_{T+1}, \quad \langle R_{T+1}, R_T^+ \rangle \leq 0.
\] (17)

Inequality (17) is known as an approachability condition, and ensures that flow regrets \(\rho_{T+1}\) point in the direction opposite to that of aggregate regrets \(R_T\). This puts strong bounds on the speed at which aggregate regrets \((R_t)_{t \geq 1}\) can grow.

By construction \((R_{2,t})_{t \geq 1}\), which measures maximum foregone gains, satisfies

\[
R_{2,T+1} = \begin{cases} 
(1 - \lambda_{T+1})(w_{T+1} - w^0_{T+1}) + R_{2,T} & \text{if } R_{2,T} \geq 0 \\
(1 - \lambda_{T+1})(w_{T+1} - w^0_{T+1}) & \text{if } R_{2,T} < 0 
\end{cases}.
\]

Hence, it follows that \(R_{2,T+1} = (1 - \lambda_{T+1})(w_{T+1} - w^0_{T+1}) + R_{2,T}^+\). Thus, condition (17) is equivalent to

\[
[\pi_{T+1} - \alpha \lambda_{T+1}(w_{T+1} - w^0_{T+1})] R_{1,T}^+ + \alpha^2 [(1 - \lambda_{T+1})(w_{T+1} - w^0_{T+1}) + R_{2,T}^+ - R_{2,T}] R_{2,T}^+ \leq 0.
\]

Noting the identity \((R_{2,T}^+ - R_{2,T})R_{2,T}^+ = 0\), it follows that approachability condition (17) is equivalent to

\[
[\pi_{T+1} - \alpha \lambda_{T+1}(w_{T+1} - w^0_{T+1})] R_{1,T}^+ + \alpha^2 [(1 - \lambda_{T+1})(w_{T+1} - w^0_{T+1})] R_{2,T}^+ \leq 0
\]

\[
\iff \quad \pi_{T+1} R_{1,T}^+ - [\alpha \lambda_{T+1} R_{1,T}^+ - \alpha^2 (1 - \lambda_{T+1}) R_{2,T}^+ (w_{T+1} - w^0_{T+1})] \leq 0.
\]

Hence approachability condition (17) can be satisfied for any realization of \(w_{T+1}\) and \(w^0_{T+1}\)

\(^{19}\) Vector \(R_T\) is defined as \((R_{1,T}, \alpha R_{2,T})\) rather than \((R_{1,T}, R_{2,T})\) only because it leads to a slight improvement in performance bounds.
by setting

$$\lambda_{T+1} = \frac{\alpha \mathcal{R}_{2,T}^+}{\mathcal{R}_{1,T}^+ + \alpha \mathcal{R}_{2,T}^+} \quad \text{and} \quad \pi_{T+1} = \begin{cases} \alpha \lambda_{T+1} (w_{T+1} - w_{T+1}^0)^+ & \text{if } \mathcal{R}_{1,T} \leq 0 \\
0 & \text{if } \mathcal{R}_{1,T} > 0 \end{cases}$$

which corresponds to the calibrated contract defined by (11) and (12).

The following lemma shows that under the contract defined by (11) and (12), incentive properties (15) and (16) are approximately satisfied. Recall that $d_t = \sup_{r \in \mathcal{R}} |\langle a_t - a_0^t, r \rangle|$ denotes the magnitude of positions taken by the agent in period $t$.

**Lemma 2** (approximate incentives). For all $T$, all $T' \leq T$ and all possible histories,

$$\Sigma_{T \setminus T'} - S_{T \setminus T'} \leq w \sqrt{\sum_{t=1}^{T} d_t^2} \leq w \bar{d} \sqrt{T} \quad (18)$$

$$-\alpha w \bar{d} \leq \Pi_T - \alpha S_T \leq \alpha w \sqrt{\sum_{t=1}^{T} d_t^2} \leq \alpha w \bar{d} \sqrt{T}. \quad (19)$$

Note that by Assumption 1, $d_t \leq \bar{d}$, so that $\sqrt{\sum_{t=1}^{T} d_t^2} \leq \bar{d} \sqrt{T}$. Hence Lemma 2 implies that incentive properties (15) and (16) hold at any possible history $h_T$, up to an error term of order $O(\sqrt{T})$. Note that this holds sample path by sample path, rather than in expectation or in equilibrium.

**Proof.** Let us first show that $||\mathcal{R}_T^+||^2 \leq \alpha^2 w^2 \sum_{t=1}^{T} d_t^2$. The proof is by induction. The property clearly holds at $T = 1$. Assume it holds at $T$. Consider the case where $\mathcal{R}_{2,T} > 0$ (i.e. there are some foregone returns). Since approachability condition (17) holds, we have that

$$||\mathcal{R}_{T+1}^+||^2 \leq ||\mathcal{R}_T^+ + \rho_{T+1}||^2 = ||\mathcal{R}_T^+||^2 + 2 \langle \mathcal{R}_T^+, \rho_{T+1} \rangle + ||\rho_{T+1}||^2$$

$$\leq ||\mathcal{R}_T^+||^2 + ||\rho_{T+1}||^2.$$
In addition $||\rho_{T+1}||^2 \leq \alpha^2 \lambda_{T+1}^2 (w_{T+1} - w_{T+1}^0)^2 + \alpha^2 (1 - \lambda_{T+1})^2 (w_{T+1} - w_{T+1}^0)^2 \leq \alpha^2 w^2 d_{T+1}^2$. Altogether this shows that the induction hypothesis holds when $R_{2,T+1} > 0$. A similar proof holds when $R_{2,T} < 0$, taking into account that in this case, $R_{2,T+1} = (1 - \lambda_{T+1}) (w_{T+1} - w_{T+1}^0)$. Hence, by induction, this implies that for all $T \geq 1$, $||R_T^+||^2 \leq \alpha^2 w^2 \sum_{t=1}^T d_t^2$. This proves (18) and the right-hand side of (19).

The left-hand side of (19) is also proven by induction. If $\Pi_T \in [\alpha S_T - \alpha w d, \alpha S_T]$, then $R_{1,T} = 0$, $\lambda_T = 1$, and $\pi_{T+1} = \alpha (w_T - w_T^0)^+$. Hence by construction, $\Pi_{T+1} \geq \alpha S_{T+1} - \alpha w d$. If instead $\Pi_T > \alpha S_T$, then by definition of $d$, $\Pi_{T+1} \geq \alpha S_{T+1} - \alpha w d$. This implies the left-hand side of (19). \[\square\]

As the next lemma shows, the approximate incentive conditions given by Lemma 2 imply performance bounds for calibrated contracts.

**Lemma 3.** Pick $\alpha_0 \in (0,1)$ and for any $\eta \in (0,1)$ let $\alpha = \alpha_0 + \eta (1 - \alpha_0)$. Consider a contract $(\lambda, \pi)$ and numbers $A, B$ and $C$ such that for all final histories $h_N$, $\Sigma_N - S_N \leq A$ and $-B \leq \Pi_N - \alpha S_N \leq C$. Then

$$r_{\lambda,\pi} \geq (1 - \eta)r_{\alpha_0} - \frac{1}{Nw} \left[ C + \frac{1}{\eta} (\alpha A + B + C) \right].$$

Theorem 2 is an immediate corollary of Lemmas 3 and 2. Intuitively, Lemma 2 shows that the calibrated contract $(\lambda, \pi)$ defined by (11) and (12) gets incentives approximately right. Lemma 3 implies that when incentives are approximately right, then performance must be approximately right as well. While this last result seems natural, it is not immediate. Whenever global incentive constraints are binding or almost binding under the benchmark linear contract of parameter $\alpha_0$, getting incentives slightly wrong may result in dismal performance. This would be the case if under the benchmark contract, the agent is indifferent between working hard and not working at all. By sharing an additional fraction $\eta$ of her returns, the principal ensures that (almost) binding global incentive compatibility constraints do not compromise performance. Madarász and Prat (2010) make the same point.
in a screening context.

A Simulation. Figure 1 illustrates the mechanics of calibrated contracts, and Lemma 2 in particular. Figure 1(a) plots a sample path for potential accumulated returns \((\Sigma_T)_{T \geq 1}\). There is significant variance and sharp drops in performance are possible. Figure 1(b) shows the induced patterns of investment shares \((\lambda_T)_{T \geq 1}\). When performance drops, investment shares diminish and when performance improves, investment shares grow. Note that shares \(\lambda_t\) are continuous rather than \(0 - 1\). This is essential for Lemma 2 to hold.\(^{20}\) As Figure 1(c) illustrates, this implies that actual excess returns \((S_T)_{T \geq 1}\) track the growth of potential returns \((\Sigma_T)_{T \geq 1}\) but do not fall as much as \((\Sigma_T)_{T \geq 1}\) when performance drops. This allows to keep tight the relationship between cumulated rewards \((\Pi_T)_{T \geq 1}\)—which are necessarily weakly increasing—and scaled actual returns \((\alpha S_T)_{T \geq 1}\). A direct implication of this, illustrated in Figure 1(d), is that the effective reward rate of the agent \((\Pi_T/S_T)_{T \geq 1}\) stays close to the target reward rate \(\alpha\). More precisely, poor performance leads to slow divergence while good performance leads to quick convergence. Because the effective reward rate is only approximately equal to \(\alpha\), this perturbs incentives a little bit and it is necessary to use a target reward rate \(\alpha\) strictly greater than \(\alpha_0\) to emulate the performance of the benchmark contract with parameter \(\alpha_0\).

4.3 Robustness to Accidents

Before turning to screening, it is worth noting an additional property of calibrated contracts: they are robust to the possibility of “accidents” during which the agent performs particularly badly over an extended amount of time. Figure 1(c) illustrates this in a striking way: whenever potential performance \((\Sigma_T)_{T \geq 1}\) drops by a significant amount, calibrated contracts significantly limit the extent of the drop in actual performance \((S_T)_{T \geq 1}\). In turn, this makes

\(^{20}\)See Foster and Vohra (1999) on the fact that \(0 - 1\) strategies cannot keep regrets small on all sample paths.
Figure 1: the behavior of calibrated contracts for a given sample path of potential returns \((\Sigma_T)_{T \geq 1}\), with target reward rate \(\alpha = 20\%\).

recovering from large performance drops possible.

This section expands on this point. Imagine that an accident can occur over some unknown time interval \([T_1, T_2]\) of arbitrary length. For instance, there may be a mistake in the agent’s trading strategy, a bias in his data. Alternatively, the agent may be temporarily irrational or have unmodeled incentives to misbehave (e.g. he is bribed to unload bad risks on the principal). Formally, this is modeled by assuming that during the random time interval
in the accident state—the agent uses an exogenously specified allocation strategy \( a^\phi_T \). This strategy may be arbitrarily bad (within the bounds imposed by Assumption 1) and need only be measurable with respect to \( F_N \). For instance, during the lapse of the accident, the agent could pick the worst ex post asset allocation in every period. Robustness to accidents of this kind is closely related to Eliaz (2002) which considers how well mechanisms perform if some players are faulty, i.e. if they use non-optimal strategies. Here, robustness to accidents corresponds to fault tolerance with respect to the agent’s selves over \([T_1, T_2]\).

In this environment, the benchmark linear contract is no longer sufficient to guarantee good performance. Accidents can undo all the profit generated by the well incentivized agent in his normal state. Strikingly, in spite of accidents, calibrated contracts are such that the excess returns generated by the agent will be approximately as high as the returns he could generate when accidents are “lucky”, i.e. when the exogenous allocation during accident states is

\[
\forall T \in [T_1, T_2], \quad a^\triangledown_T = \begin{cases} 
  a^0_T & \text{if } \sum_{t=T_1}^{T_2} w^\triangledown_t - w^0_t < 0 \quad \text{(accident is unlucky)} \\
  a^\triangledown_T & \text{if } \sum_{t=T_1}^{T_2} w^\triangledown_t - w^0_t > 0 \quad \text{(accident is lucky)}
\end{cases}
\]

where \( w^\triangledown_t \) is the realized wealth under the \( a^\triangledown_t \) at time \( t \). Denote by \( r^\triangledown_{\lambda, \pi} \) the net expected returns to the principal when accidental behavior is \( a^\triangledown_t \) and the calibrated contract is used. Denote by \( r^\triangledown_{\alpha} \) the net expected returns to the principal when accidental behavior is \( a^\triangledown_t \) and the benchmark contract of parameter \( \alpha \) is used. The following holds.

**Theorem 3** (accident proofness). Pick \( \alpha_0 \) and for any \( \eta > 0 \), set \( \alpha = \alpha_0 + \eta(1 - \alpha_0) \). There exists a constant \( m \), independent of \( N \) and \( P \) such that,

\[
   r^\triangledown_{\lambda, \pi} \geq (1 - \eta) r^\triangledown_{\alpha_0} - \frac{m}{\sqrt{N}}.
\]
5 Screening

As it is, the calibrated contract defined by (11) and (12) does not induce untalented managers
to screen themselves. Rewards are positive, and a sufficiently long-lived uninformed agent can
obtain large expected payoffs from luck and volatility alone. Indeed, imagine that the agent
has no information and all assets have the same expected returns. By systematically picking
assets different from the benchmark allocation, the agent ensures that \((\Sigma_T)_{T \geq 1}\) is a martingale
with volatility bounded away from 0. Hence, under appropriate time normalization, \((\Sigma_T)_{T \geq 1}\)
behaves like a Brownian motion.\(^{21}\) Lemma 2 implies that the agent’s payoff in period \(T\)
satisfies \(\Pi_T \geq \alpha \Sigma_T - \alpha \overline{d} (1 + \sqrt{T})\). In addition, since \(\Pi_T\) is weakly increasing in \(T\), it
follows that \(\Pi_N \geq \max_{T \leq N} \Sigma_T - \alpha \overline{d} (1 + \sqrt{N})\). Hence, given that \(\max_{T \leq N} \Sigma_T\) behaves
approximately like the maximum of a Brownian motion, the agent can obtain rewards of
order \(\sqrt{N}\) with non-vanishing probability.

A simple modification of the contract described by (11) and (12) achieves screening by
imposing an initial participation cost \(-b\) and only paying the agent when his performance
is above a dynamic hurdle \(\Theta_T\) which depends on the size of positions he has been taking.

Given a free parameter \(M > 0\), define

\[
\Theta_T \equiv 2w \left( 1 + \sqrt{d^2 + \sum_{t=1}^{T} \lambda_t^2 d_t^2} \right) \sqrt{M + \ln \left( \frac{d^2 + \sum_{t=1}^{T} \lambda_t^2 d_t^2}{M + \ln T} \right)},
\]  

(20)

where \(d_t = \sup_{r_t \in R} |\langle a_t - a_t^0, r_t \rangle|\) and \(\lambda_t d_t\) measures the size of the agent’s effective bet
\(\lambda_t (a_t - a_t^0)\) away from the default allocation \(a_t^0\) (note that by Assumption 1, \(d_t \leq \overline{d}\)). Hurdle
\(\Theta_T\) is an aggregate measure of how active the agent has been. If the agent makes significant
bets away from \(a_t^0\) in every period then \(\Theta_T\) will be of order \(\sqrt{T \ln T}\). If the agent makes few
bets, hurdle \(\Theta_T\) will remain small. The quantity \(d^2 + \sum_{t=1}^{T} \lambda_t^2 d_t^2\) is a measure of time under
which \((\Sigma_T)_{T \geq 1}\) will have at most the variation of a standard Brownian motion.

Hurdled calibrated contracts are defined by a sequence \((\lambda_t, \pi_t, \pi_t^\Theta)_{t \geq 1}\). The sequence

\(^{21}\)See, for instance, Billingsley (1995), Theorem 35.12.
$(\lambda_t, \pi_t)_{t \geq 1}$ is still defined according to recurrence equations (11) and (12), and $\lambda_t$ is still the share of wealth actually invested by the agent. However, for $t \geq 1$, reward $\pi_t$ is no longer paid to the agent for sure. Rather, the agent is paid a hurdled reward $\pi^\Theta_t$ such that $\pi^\Theta_1 = -b$, and $\pi^\Theta_t = 1_{S_t \geq \Theta_t} \pi_t$, i.e. potential reward $\pi_t$ is paid to the agent if and only if $S_t \geq \Theta_t$.

This hurdled contract coincides with the baseline calibrated contract defined in Section 4, except that: (i) the agent must pay a participation fee $-b$ in the first period, and (ii) the agent obtains rewards only when actual performance $S_T$ is above a hurdle $\Theta_T$ which grows at a rate at most $\sqrt{T \ln T}$. Theorem 4 will show that this contract induces uninformed agents to screen themselves in the first period, and imposes only a moderate incentive cost on informed agents.

An intuitive rationale for the form of hurdle $\Theta_T$ is as follows. Let the agent be uninformed, so that the process $(S_t)_{t \geq 1}$ is at best a martingale, and imagine that the agent is frequently active, i.e. $\sum_{t=1}^{T} d_t^2$ is of order $T$. Then hurdle $\Theta_T$ is of order $\sqrt{T \ln T}$. The law of the iterated logarithm implies that with probability 1, as $T$ gets large, $\max_{T' \leq T} S_{T'}$ is of order $\sqrt{T \ln \ln T}$. Because $\frac{\sqrt{T \ln \ln T}}{\sqrt{T \ln T}}$ goes to 0 as $T$ grows large, hurdle $\Theta_T$ insures that uniformed agents have very little hope to obtain unjustified returns. Indeed, the following result holds.

**Lemma 4** (hurdle effectiveness). If the agent is uninformed, then for any allocation strategy $a$,

$$E_a \left( \sum_{t=1}^{N} 1_{S_t \geq \Theta_t} \right) \leq \frac{\pi^2}{2} \exp(-2M),$$

where $\pi$ is the constant 3.1415…

As the next lemma shows, the use of hurdles comes only at a moderate incentive cost. Let $\Pi^\Theta_T = \sum_{t=1}^{T} \pi^\Theta_t$ denote actual rewards, up to time $T$.

22See Billingsley (1995), Theorem 9.5.
Lemma 5 (approximate incentives). For all $T, T' < T$, and all paths of play, we have that

$$\Sigma_{T \setminus T'} - S_{T \setminus T'} \leq w \sqrt{T} \sum_{t=1}^{T} d_t^2$$

(21)

$$-\alpha \Theta_T - \alpha w d - b \leq \Pi_T^0 - \alpha S_T \leq \alpha w \sqrt{T} \sum_{t=1}^{T} d_t^2.$$  

(22)

Combining Lemmas 3, 4 and 5 yields the main result of this section. Denote by $r_{\lambda, \pi^\Theta}$ the net expected per-period returns generated by the agent under the hurdled calibrated contract.

Theorem 4 (performance with screening). Pick $\alpha_0 \in (0, 1)$ and for any $\eta \in (0, 1)$, let $\alpha > \alpha_0 + \eta(1 - \alpha_0)$. There exists a constant $m$ independent of time horizon $N$ and probability space $\mathcal{P}$ such that for all $h_T$,

$$r_{\lambda, \pi^\Theta}|h_T \geq (1 - \eta) r_{\alpha_0}|h_T - m \sqrt{\ln N \over N}$$

(23)

Furthermore, whenever $-b + \alpha w \bar{d} \times \frac{\pi^*}{2} \exp(-2M) < 0$, it is strictly optimal for uninformed agents not to participate.

The combination of initial fee $-b$ and hurdle $\Theta_t$ induces early screening by uninformed agents. Hurdle $\Theta_t$ is large enough that uninformed agents have little hope to be rewarded by luck but small enough that it does not significantly affect the incentives of informed agents. The penalty which was of order $\frac{1}{\sqrt{N}}$ in Theorem 2 is now of order $\sqrt{\ln N \over N}$. The next lemma provides conditions under which the performance loss from screening is in fact of order $\frac{1}{\sqrt{N}}$.

Assumption 2 (grainy returns). Let $(c, a^*)$ denote the agent’s policy under the benchmark contract with rate $\alpha_0$. There exists $\xi > 0$ such that whenever $\mathbb{E}_{c,a^*}[w_t - w_t^0|\mathcal{F}_t] > 0$, then $\mathbb{E}_{c,a^*}[w_t - w_t^0|\mathcal{F}_t] > \xi$. 

29
Theorem 5. Pick \( \alpha_0 \) and for any \( \eta > 0 \), set \( \alpha = \alpha_0 + \eta(1 - \alpha_0) \). If Assumption 2 holds, there exists a constant \( m \) such that for all \( N \) and all probability spaces \( \mathcal{P} \),

\[
r_{\lambda, \pi, \Theta} \geq (1 - \eta)r_{\alpha_0} - m \frac{1}{\sqrt{N}}.
\]

Indeed, whenever Assumption 2 holds, it can be shown that hurdles grow at a slow rate compared to expected excess returns. In fact the expected number of payments that are omitted because of hurdles is bounded above independently of \( N \).

6 Discussion

6.1 Relation to High-Watermark Contracts

High-watermark contracts. The calibrated contracts described in Sections 4 and 5 are closely related to the high-watermark contracts frequently used in the financial industry (see for instance Goetzmann et al. (2003) who develop an option-pricing approach to high-watermark contracts, or Panageas and Westerfield (2009) who show that they need not lead to excessive risk-taking when agents have infinite horizons). High-watermark contracts are structured as follows: at time \( T \), the investment share \( \lambda_T \) is always 1, and the agent gets paid

\[
\pi_T^{\text{wmk}} = \alpha \left( \sum_{t=1}^{T} w_t - w_t^0 - \max_{T' < T} \left[ \sum_{t=1}^{T'} w_t - w_t^0 \right] \right)^+.
\]

(24)

Quantity \( \max_{T' < T} \left[ \sum_{t=1}^{T'} w_t - w_t^0 \right] \) represents the maximum historical returns at time \( T \) — i.e the high-watermark. The agent only gets paid when he improves on his own historical performance. Note that high-watermark contracts are dynamic and satisfy limited liability constraints (4) and (5). In particular, for all \( T \), \( \pi_T \in [0, \alpha w_d] \).

High-watermark contract, as well as calibrated contracts, attempt to reward the agent a share \( \alpha \) of his externality on the principal. In other words, both types of contracts attempt to
keep aggregate rewards $\Pi_T$ close to $\alpha S_T$. Lemma 2 shows that calibrated contracts achieve this goal for any realization of uncertainty and any allocation strategy. High-watermark contracts also do well in this respect, provided that the sequence of returns does not have prolonged downturns. For instance, if an agent has performance $(1, 1, 1, 1, \cdots)$, then $S_N = N$ and $\Pi_{wmk}^N = \alpha N = \alpha S_N$. If the agent has performance $(1, -1, 1, -1, \cdots)$, then $S_N \in \{0, 1\}$ and $\Pi_{wmk}^N = \alpha$, so that $\Pi_{wmk}^N = \alpha S_N + o(N)$. In this respect, high watermark contracts do much better than static option contracts of the form $\pi_t = \alpha (w_t - w_0^t)^+$, which would give the agent a reward of order $\alpha N/2$.

**The value of jointly calibrating rewards and investment shares.** An important issue with high-watermark contracts—which calibrated contracts resolve—is that they do not behave well if the agent’s performance goes through an extended downturn. This is illustrated by Figure 2(a). Whenever there is an extended drop in performance, the relationship between rewards $\Pi_T$ and performance $\alpha S_T$ breaks down. Indeed $\Pi_T$ is by construction weakly increasing while under the high-watermark contract, $\alpha S_T$ can decrease in arbitrary ways. This has two implications. First, because the agent does not suffer from extended downturns, an agent who has lost the ability to generate positive return (e.g. his information has become unreliable) may cause large losses by choosing negative expected value investments that generate variance. Second, if a talented agent has been unlucky and experienced a drop in returns, the difficulty of catching up with a high watermark may discourage investment altogether.

As Figure 2(b) illustrates, calibrated contracts ultimately boil down to writing a high-watermark contract on the modified performance measure $S_T$. By choosing investment shares $\lambda_t$ appropriately, calibrated contracts are able to keep tight the relationship between rewards $\Pi_T$ and actual performance $S_T$ at every history. As a result, extended downturns have a much more limited impact on incentives. As was noted previously, investment shares $\lambda_t$ must move smoothly with performance. Rather than a stop-loss provision, it is more accurate to think of the calibrated investment shares $(\lambda_t)_{t\geq 0}$ as continuously implementing
a robust option on the agent’s potential performance $\Sigma_T$.

Figure 2: high-watermark and calibrated contracts for a sample path of potential returns $(\Sigma_T)_{T \geq 1}$, with target reward rate $\alpha = 20\%$.

**Connection with optimal contracting.** It is instructive to note that for sufficiently high discount factors, DeMarzo and Sannikov (2006) as well as Biais et al. (2007, 2010) also derive high-watermark contracts as optimal contracts in their specific environments. The link is not entirely obvious because they follow the usual approach of characterizing the optimal contract through movements in continuation values, i.e. the optimal contract is described in the standard (forward looking) language of continuation values. Because calibrated (and high-watermark) contracts are detail-free, they can only be described in reference to (backward looking) realized observables. To a first order, DeMarzo and Sannikov (2006) and Biais et al. (2007, 2010) find that in their environment, under optimal contracts, the agent’s continuation value follows a Brownian motion, proportional to the agent’s performance, and reflected at some upper bound $W$. Whenever the agent’s continuation value hits this upper bound, he is paid a proportion of his upside. This in fact encodes for a high-watermark contract. Imagine that at time $t$, the agent is promised value $\bar{W}$, and that he starts losing money. Then, his

---

23 See DeMarzo et al. (2009) for work on the relation between approachability and robust option pricing.
continuation value moves in a way proportional to his performance, and he is only paid again when his performance covers his losses so that his continuation value climbs back to \( \overline{W} \). This coincides with the reward profile of a high-watermark contract: the agent only gets paid once he has recouped his losses. This connection should not be entirely surprising: DeMarzo and Sannikov (2006) as well as Biais et al. (2007, 2010) consider environments with linear production technology in which the benchmark high-liability contracts of Section 3 are essentially optimal; calibrated contracts are specifically designed to approximate the performance of such contracts.

The connection is particularly strong with Biais et al. (2007) and especially Biais et al. (2010) who emphasize the role of downsizing the size of the project managed by the agent as a function of his performance. This is related to varying investment shares \((\lambda_t)_{t \geq 1}\) in the current paper. The use of downsizing in Biais et al. (2007, 2010) however is slightly different. In their work, downsizing occurs when continuation values are so low that at the current size of the project, optimal behavior can no longer be enforced. Downsizing allows to deliver the promised low values while maintaining appropriate incentive compatibility conditions. As a result, downsizing occurs only after sufficiently long strings of poor performance. In the current paper, \((\lambda_t)_{t \geq 0}\) can be seen as a preventive downsizing scheme, which rules out continuation values so low that incentive provision becomes problematic.

**Screening.** High-watermark contracts do not achieve screening. The contracts described in this paper achieve screening by using hurdles that link payoffs to the size of positions that the agent has been taking. Given an initial participation cost \(-b\), these activity based hurdles insure that it is not tempting for uninformed agents to participate. Note that these hurdles depend only on activity measure \(d_t = \sup_{r_t \in \mathbb{R}} | \langle a_t - a_0^t, r_t \rangle |. \) In particular the actual asset allocation chosen by the agent need not be made public, which matters if the agents are worried about privacy.

In addition note that the use of an accurate counter-factual performance measure \(w_0^t\) is essential for the success of the screening strategy. Indeed, it ensures that whenever the
agent is uninformed, then $\Sigma_T = \sum_t^T w_t - w_0^t$ is a martingale with weakly negative drift (i.e. a surmartingale). Not all contracts have this property. Imagine for instance that an ex ante counter factual performance measure, such as $\bar{w}_t^0 = \mathbb{E}(w_t^0 \mid \mathcal{F}_1^0)$, is used. Because, $\Sigma_T = \sum_{t=1}^T w_t - w_0^t$ is not a martingale with weakly negative drift, there can be non-vanishing probability that a fixed asset allocation beat period 1 expectations $\bar{w}_0^0$ arbitrarily many times. As a result screening cannot be achieved using ex ante counterfactual performance measures.

**Absent Features.** Finally, a surprising property of the calibrated contracts developed in Sections 4 and 5 is that they do not require the use of contractual provisions often deemed necessary to align the interests of the principal and the agent: agents are not required to hold a significant amount of the asset allocation they are suggesting and there is no use of clawbacks or deferred payments. Of course this doesn’t mean that such provision aren’t useful, especially if the agent’s horizon is small.

### 6.2 Future Work

This paper develops a robust approach to dynamic contracting in two steps: the first step identifies high-liability linear contracts that satisfy attractive efficiency properties regardless of the underlying environment; the second step shows how to approximate the performance of benchmark contracts using limited-liability dynamic contracts. The contracting strategy is to calibrate rewards to the agent as well as the share of wealth he manages, so that key properties of the benchmark contract are approximately replicated. The resulting calibrated contracts are simple, and perform approximately as well as an attractive benchmark under very general conditions. The simplicity of the results is encouraging and suggests that the approach might be fruitfully applied in other settings.

From a theoretical perspective, a first valuable extension would be to allow for risk-aversion on both sides. Appendix A develops extensions to environments where the principal is risk averse or the agent may have varying marginal utility of income, but more work remains.
to be done on that issue. For instance, the calibration strategy described in Section 4 results in unnecessary variation in the rewards to the agent: if he has been performing well, but underperforms one period, he does not receive rewards in the following period. When the agent is risk-averse, such a calibration strategy leads to inefficiencies and smoother calibration techniques become desirable.

Another avenue for research is to develop robust approaches to pick an appropriate reward rate $\alpha$ for the benchmark linear contract. In this respect, it may be fruitful to consider multi-agent mechanisms that may help extract information from agents. Considering multi-player environments may also be interesting beyond the principal-agent setting that this paper focuses on. For instance many attractive allocation mechanisms, such as VCG, require agents to make significant payments and are therefore ill-suited in environments where agents are severely cash constrained. A dynamic calibration approach such as the one developed in this paper may help relax such limited liability constraints.

Finally, with actual implementation in mind, it seems important to determine whether calibrated contracts really do induce approximately good behavior from agents. Indeed, Theorems 2 and 4 may rely excessively on the agent’s ability to understand the incentive properties of calibrated contracts. This is ultimately an empirical question. An advantage of the calibrated contract approach is that it lends itself naturally to realistic experiments using actual returns data, since the contracts should perform well regardless of the agent’s beliefs over the process for returns.

Appendix

A Additional Results and Extensions

This appendix presents a number of additional results: Appendix A.1 shows that static limited liability contracts cannot approximate the performance of benchmark contracts;
Appendix A.2 considers principal-agent problems where the set of actions is not convex; Appendix A.3 tackles risk-aversion on the side of the principal; Appendix A.4 allows for varying levels of wealth; Appendix A.5 derives rough performance bounds in the case where the agent’s preferences are misspecified.

A.1 Static Contracts under Limited Liability

This appendix shows that there is no limited-liability static contract that can approximate the performance of the high-liability benchmark contract described in Section 3 (even though the benchmark contract is itself static). Option-like static contracts that reward the agent according to $\alpha(w_t - w_0^t)$ have well-known issues: uninformed agents can obtain large payoffs, and talented agents may be induced to choose asset allocations with large variance and negative expected value. All static limited-liability contracts suffer from similar issues.

Consider contracts such that for all $t \geq 1$ reward $\pi_t$ depends only on returns at time $t$ (i.e. $\pi_t = \pi_t(w_t, w_0^t)$) and $\pi_t \in [-b, w_t]$. Note that this is actually a weaker limited liability constraint than that imposed by conditions (4) and (5). The following lemma uses an argument similar to that of Foster and Young (2010) to show that such contracts cannot simultaneously screen agents and reward them a share of the surplus they create. The proof requires that the ex post optimal asset allocation be uncertain from the perspective of public information, i.e. any selection $a_t \in \arg\max_{a \in A} \langle a, r_t \rangle$ is not $\mathcal{F}_t^0$-measurable.

Lemma A.1. Consider a one-shot reward function $\pi_t$ satisfying limited liability, and such that for all distributions $w_t$ of realized wealth,

$$
\mathbb{E}\pi_t \geq \alpha \mathbb{E}[w_t - w_0^t].
$$

There exists $\alpha > 0$ such that for $w$ large enough, an untalented manager can obtain expected profit at least $\alpha w$.

Proof. Let $\overline{a}_t$ denote an ex post optimal asset allocation, i.e. $\overline{a}_t \in \arg\max_{a \in A} \langle a, r_t \rangle$. By
assumption, we have that \( \bar{\tau} \equiv \mathbb{E} \langle \bar{a}_t - a_{0t}, r_t \rangle > 0 \). Since \( A \) is a compact set, for \( M \) large enough, there exists \( M \) points \( \{a_1, \cdots, a_M\} \subset A \) such that for all \( a \in A \), there exists \( k \in \{1, \cdots, M\} \) satisfying for all \( r_t \), \( |a_k - a, r_t| \leq \bar{\tau}/2 \). Hence there is a selection \( \bar{\pi}_t^M \) of assignments in \( \{a_1, \cdots, a_M\} \) such that \( \mathbb{E} \langle \bar{a}_t^M - a_{0t}, r_t \rangle > \bar{\tau}/2 \).

Consider the allocation strategy \( \bar{a} \), consisting picking an allocation \( a \in \{a_1, \cdots, a_M\} \) at random, and with equal probabilities. The agent’s expected payment satisfies

\[
\mathbb{E}_{\bar{a}} \pi_t \geq \frac{1}{M} \mathbb{E}_{a_t^M} \pi_t - b \geq \frac{1}{2M} \alpha \bar{\tau} w - b.
\]

This concludes the proof. \( \square \)

A corollary of this is that if the agent receives significant rewards using static limited liability contracts, there can be no screening. Such misaligned incentives can also affect talented managers. Imagine that at each time \( t \) there is positive probability that the information \( I_c^t \) acquired by the manager is worthless: the optimal portfolio is the same under \( \mathcal{F}_t \) and \( \mathcal{F}_t^0 \). When this happens, the manager cannot deliver excess returns. However, by deviating from truthtelling, the manager can obtain an expected payoff of at least \( \alpha w \).

### A.2 Principal-Agent Problems Without a Convex Action Space

This appendix extends the analysis to a principal-agent framework more general than the financial contracting problem studied in the paper. The principal and the agent are still risk-neutral, but in every period the agent suggests and implements an action \( a_t \in A \), where \( A \) is a potentially non-convex set of actions. Every period, a state of the world \( r_t \) is drawn which, given action \( a \), yields observable payoffs \( w(a, r_t) \) to the principal. Cost \( c_t \) may now represent the cost of information acquisition, as well as the cost of making a specific action available. Action \( a_{0t}^t \) is the action that the principal would (could) implement on her own. The main difference is that because set \( A \) need not be convex, the principal must use randomized strategies to calibrate her contract with the agent.
The calibrated contract of Section 4 can be adapted as follows. Parameter $\lambda_t$ now denotes the probability that the principal follow the action suggested by the agent.\footnote{For calibration results to hold, it is important that the agent not be able to condition his suggested action on the outcome of the principal’s randomization. If the agent takes the action on behalf of the principal, $\lambda_t$ should be interpreted as the probability that the principal approve the agent’s proposed course of action.} Let $a_t^\lambda$ denote the action actually taken at time $t$. Denote by $\psi_t \equiv w(a_t, r_t) - w(a_0^0, r_t)$ the potential excess returns and by $\psi_t^\lambda \equiv w(a_t^\lambda, r_t) - w(a_0^0, r_t)$ the realized excess returns. As in Section 4, define

$$\Sigma_T = \sum_{t=1}^T \psi_t, \quad \Pi_T = \sum_{t=1}^T \pi_t, \quad S_T = \sum_{t=1}^T \psi_t^\lambda,$$

as well as $\Sigma_{T \setminus T'} = \Sigma_T - \Sigma_{T'-1}$, $\Pi_{T \setminus T'} = \Pi_T - \Pi_{T'-1}$ and $S_{T \setminus T'} = S_T - S_{T'-1}$. As in Section 4, regrets are defined by

$$R_{1,T} = \Pi_T - \alpha S_T \quad \text{and} \quad R_{2,T} = \max_{T' \leq T} \Sigma_{T \setminus T'} - S_{T \setminus T'}.$$

Let $\mathcal{R}_T = (R_{1,T}, \alpha R_{2,T})$. Contract $(\lambda_t, \pi_t)_{t \in \mathbb{N}}$ is defined by

$$\lambda_{T+1} = \frac{\alpha R_{2,T}^+}{R_{1,T}^+ + \alpha R_{2,T}^+} \quad \text{and} \quad \pi_{T+1} = \begin{cases} 0 & \text{if} \quad R_{1,T} > 0 \\ \alpha \psi_{T+1}^\lambda & \text{if} \quad R_{1,T} \leq 0 \end{cases} \quad (25)$$

with the convention that $0/0 = 1$. Lemma 2 extends as follows.

**Lemma A.2** (approximate incentives). For all $T$, and any strategy $(c, a)$ of the agent, we have that

$$\mathbb{E}_{c,a} \Sigma_T - \mathbb{E}_{c,a} S_T \leq w \bar{d} \sqrt{T} \quad (26)$$

$$-\alpha w \bar{d} \leq \Pi_T - \alpha \mathbb{E}_{c,a} S_T \leq \alpha w \bar{d} \sqrt{T}. \quad (27)$$

\textbf{Proof.} The left-hand side of (27) follows from a proof identical to that of the left-hand side of (19).
Let us turn to the other inequalities. Let $\rho_T = R_T - R_{T-1}$ denote flow regrets, and observe that $E_{c,a} \left( \langle R_{T-1}^+, \rho_T \rangle \right) \leq 0$. Hence, a proof identical to that of Lemma 2 yields that

$$\forall (c,a), \quad E_{c,a} \| R_T^+ \|^2 \leq (\alpha d)^2 T.$$ 

Finally note that by Jensen’s inequality, for all $i \in \{1,2\}$,

$$E_{c,a}(R_{i,t}^+) \leq E_{c,a} \left( \sqrt{\| R_{i,t}^+ \|^2} \right) \leq \sqrt{E_{c,a} \left( \| R_{i,t}^+ \|^2 \right)} \leq \alpha w d \sqrt{T}. $$

This implies (26) and the right-hand side of (27).

Given Lemma A.2, a proof identical to that of Lemma 3 yields the following performance bound: pick $\alpha_0, \eta > 0$ and let $\alpha = \alpha_0 + \eta(1 - \alpha_0)$. There exists $m$ independent of $N$ and $\mathcal{P}$ such that

$$r_{\lambda,\pi} \geq (1 - \eta)r_{\alpha_0} - \frac{m}{\sqrt{N}}.$$

### A.3 Risk Averse Principal

The paper focuses on the case where both the principal and the agent have quasilinear preferences. A full-fledged analysis of the case where both the principal and the agent can be risk averse is beyond the scope of the paper. This section provides a partial extension to the case where the agent is risk neutral while the principal is risk averse.

Consider an increasing concave utility function $u$. This section considers the case where the agent is risk-neutral while the principal has utility function $u$ over flow wealth. This section shows how to construct calibrated contracts such that payoffs to the agent and residual utility to the principal satisfy

$$\sum_{t=1}^{N} \pi_t = \nu \sum_{t=1}^{N} \left[ u(w_t - \pi_t) - u(w^0_t) \right] + o(N), \quad (28)$$

where $\nu > 0$ is a design parameter used to shift surplus between the principal and the agent.
If this condition holds, it ensures that whenever the agent gets positive surplus, the principal must obtain a commensurate expected payoff.\textsuperscript{25} Note that if \( u(w) = w \), then condition (28) simply boils down to the condition that \( \sum_{t=1}^{N} \pi_t = \alpha \sum_{t=1}^{N} w_t - w_0 + o(N) \) with \( \alpha = \nu/(1+\nu) \), i.e. the case already treated in the paper.

**Preliminaries.** Let us begin by providing a generalization of Assumption 1.

**Assumption 3.** There exists \( \overline{d} \) such that for all \( (a, a') \in A^2 \) and all \( r_t \in R \),

\[
|u(w(1 + \langle a, r_t \rangle)) - u(w(1 + \langle a', r_t \rangle))| \leq \overline{d}.
\]

In addition, it is assumed that there exists \( \kappa > 0 \) such that for any possible realized wealth \( w_t, u'(w_t) \leq \kappa \). Let \( \phi(\cdot, \cdot) \) denote the implicit function uniquely defined by

\[
\forall w_1, w_0, \quad \phi(w_1, w_0) = \nu \left[ u(w_1 - \phi(w_1, w_0)) - u(w_0) \right],
\]

(29)

Note that by construction, \( |\phi(w_t, w_0^t)| \leq \nu \overline{d} \). The following properties will be useful in the analysis.

**Lemma A.3.**

(i) \( \phi(w, w) = 0 \) for all \( w \);

(ii) \( \phi(\cdot, \cdot) \) is increasing and concave in its first argument;

Proof. Point (i) and the fact that \( \phi \) is increasing in its first argument follow immediately from (29). The fact that \( \phi(w_1, w_0) \) is concave in \( w_1 \) follows from concavity of \( u \). For any values, \( w_0, w_1, w_2 \) and \( \rho \in (0, 1) \), let us define

\[
\phi_1 = \nu[u(w_1 - \phi_1) - u(w_0)]; \quad \phi_2 = \nu[u(w_2 - \phi_2) - u(w_0)]
\]

\[
\phi_\rho = \rho \phi_1 + (1 - \rho) \phi_2 \quad \text{and} \quad w_\rho = \rho w_1 + (1 - \rho) w_2.
\]

\textsuperscript{25}An important motivating example for this extension is the case in which \( u = \log \) and \( w_t^0 = w(1 + \langle a_t^0, r_t \rangle) \), with \( a_t^0 \in \arg \max_{a_t \in A} E[\log[w(1 + \langle a, r_t \rangle)]|\mathcal{F}_t^0] \). The corresponding calibrated contract would be appropriate if wealth is accumulated (with compounded returns) and the principal has log utility over the final outcome.
By concavity of $u$ we have that
\[
\nu[u(w_\rho - \phi_\rho) - u(w_0)] \geq \nu[\rho u(w_1 - \phi_1) + (1 - \rho)u(w_2 - \phi_2) - u(w_0)] \\
\geq \rho \phi_1 + (1 - \rho)\phi_2 = \phi_\rho.
\]

Hence, it must be that $\phi(w_\rho, w_0) \geq \phi_\rho = \rho \phi(w_1, w_0) + (1 - \rho)\phi(w_2, w_0)$, i.e. $\phi(\cdot, \cdot)$ is concave in its first argument.

**Calibrated contracts.** For any sequence of adapted investment shares $\lambda = (\lambda_t)_{t \geq 0}$ and actual payments $(\pi_t)_{t \geq 0}$, let $w^\lambda_t = \lambda_t w_t + (1 - \lambda_t)w^0_t$, and
\[
\Sigma_T = \sum_{t=1}^T \nu[u(w_t - \pi_t) - u(w^0_t)]; \quad \Pi_T = \sum_{t=1}^T \pi_t; \quad S_T = \sum_{t=1}^T \nu[u(w^\lambda_t - \pi_t) - u(w^0_t)]. \tag{30}
\]

For any $T' < T$, let $\Sigma_{T' \setminus T'} = \Sigma_T - \Sigma_{T'-1}$, $\Pi_{T' \setminus T'} = \Pi_T - \Pi_{T'-1}$ and $S_{T' \setminus T'} = S_T - S_{T'-1}$. In addition, let
\[
\mathcal{R}_{1,T} = \Pi_T - S_T \quad \text{and} \quad \mathcal{R}_{2,T} = \max_{T' < T} [\Sigma_{T' \setminus T'} - S_{T' \setminus T'}] \tag{32}
\]

The objective is to calibrate payments $(\pi_t)_{t \geq 0}$ and investment shares $(\lambda_t)_{t \geq 0}$ so that $\mathcal{R}_{1,T}$ and $\mathcal{R}_{2,T}$ remain small compared to $T$.

As in Section 4, this can be achieved by setting
\[
\lambda_{T+1} = \frac{\mathcal{R}_{2,T}^-}{\mathcal{R}_{1,T}^- + \mathcal{R}_{2,T}^-} \quad \text{and} \quad \pi_{T+1} = \begin{cases} 0 & \text{if } \mathcal{R}_{1,T} > 0 \\ \phi(w^\lambda_{T+1}, w^0_{T+1})^+ & \text{if } \mathcal{R}_{1,T} \leq 0 \end{cases} \tag{31}
\]

Under this calibrated contract the following extension of Lemma 2 holds.

**Lemma A.4.** For all $T$ and $T' < T$, we have that
\[
\Sigma_{T' \setminus T'} - S_{T' \setminus T'} \leq \nu d \sqrt{T} \tag{32}
\]
\[
-\nu d \leq \Pi_T - S_T \leq \nu d \sqrt{T}. \tag{33}
\]
Proof. The first part of the proof shows that as $T$ grows large, $||R^\perp_T||$ remains small compared to $T$. Let $R_T = (R_{1,T}, R_{2,T})$ and $\rho_T = R_T - R_{T-1}$. We have that

$$R_{2,T+1} = \nu [u(w_{T+1} - \pi_{T+1}) - u(w_{T+1}^0) - [u(w_{T+1}^\lambda - \pi_{T+1}) - u(w_{T+1}^0)]] + R^\perp_{2,T,1}.$$  

Because $R_{2,T}(R_{2,T} - R_{2,T}) = 0$, it follows that

$$\langle R^\perp_T, \rho_{T+1} \rangle = R^\perp_{1,T,\pi_{T+1}^T} + \nu (R^\perp_{2,T} [u(w_{T+1} - \pi_{T+1}) - u(w_{T+1}^0)])$$

$$- [R^\perp_{1,T} + R^\perp_{2,T} [u(w_{T+1}^\lambda - \pi_{T+1}) - u(w_{T+1}^0))].$$

We want to show that $\langle R^\perp_T, \rho_{T+1} \rangle \leq 0$. Consider first the case where $\pi_{T+1} = 0$. Given concavity of $u$, straightforward algebra yields that

$$\langle R^\perp_T, \rho_{T+1} \rangle \leq R^\perp_{1,T,\pi_{T+1}^T} + \nu (R^\perp_{2,T} [u(w_{T+1} - \pi_{T+1}) - u(w_{T+1}^0)])$$

$$\leq 0,$$

where the last inequality follows from the fact that $(\lambda_t, \pi_t)_{t \geq 0}$ satisfies (31). Now consider the case where $\pi_{T+1} > 0$. By construction, we must have that $w_{T+1} > w_{T+1}^0$ and $\pi_{T+1} = \phi(w_{T+1}^\lambda, w_{T+1}^0)$. By definition of $\phi$ and using the concavity of $\phi$ in its first argument (Lemma A.3), as well as the fact that $(\lambda_t, \pi_t)_{t \geq 0}$ satisfies (31), we obtain that

$$\langle R^\perp_T, \rho_{T+1} \rangle = R^\perp_{1,T,\pi_{T+1}^T} + R^\perp_{2,T} \phi(w_{T+1}, w_{T+1}^0) - [R^\perp_{1,T} + R^\perp_{2,T}] \phi(w_{T+1}^\lambda, w_{T+1}^0)$$

$$\leq R^\perp_{1,T,\pi_{T+1}^T} + (R^\perp_{2,T} - \lambda_{T+1} [R^\perp_{1,T} + R^\perp_{2,T}]) \phi(w_{T+1}, w_{T+1}^0) \leq 0.$$  

We now prove by induction that $||R^\perp_T||^2 \leq (\nu d)^2 T$. The property clearly holds for $T = 1$. Assume that it holds for $T \geq 1$ and let us show it must hold for $T+1$. Consider first the
case where $R_{2,T} > 0$. We have that

$$||R^+_{T+1}||^2 \leq ||R^+_{T}||^2 + 2 \langle R^+_{T}, \rho_{T+1} \rangle + ||\rho_{T+1}||^2 \leq ||R^+_{T}||^2 + (\nu d)^2$$

(34)

where we used the fact that $\langle R^+_{T}, \rho_{T+1} \rangle \leq 0$, and

$$||\rho_{T+1}||^2 \leq \nu^2 \left( [u(w^*_{T+1} - \pi_{T+1}) - u(w^0_{T+1})]^2 - 2[u(w_{T+1} - \pi_{T+1}) - u(w^0_{T+1})][u(w^*_{T+1} - \pi_{T+1}) - u(w^0_{T+1})] \right) \leq \nu^2 [u(w_{T+1} - \pi_{T+1}) - u(w^0_{T+1})]^2.$$

The last inequality uses the fact that $w^*_{t} - \pi_{t}$ is always bracketed by $w_{t} - \pi_{t}$ and $w^0_{t}$. Altogether, this implies that the induction hypothesis holds for $T + 1$ when $R_{2,T} > 0$. A similar proof holds in the case where $R_{2,T} \leq 0$. Hence, for all $T \geq 1$, $||R^+_{T}||^2 \leq (\nu d)^2 T$.

Inequality $||R^+_{T}|| \leq \nu d \sqrt{T}$ implies (32) and the right-hand side of (33). The left-hand side of (33) follows from a proof identical to that of the left-hand side of (19).

Let us denote by $r_{\lambda,\pi} = \frac{1}{N} E_{\lambda,\pi} \left( \sum_{t=1}^{N} u(w^*_{t} - \pi_{t}) - u(w^0_{t}) \right)$ the average expected utility gain when the agent is given the calibrated contract of parameter $\nu$. Similarly, denote by $r_{\text{max}} = \frac{1}{N} E_{\pi,a^*} \left( \sum_{t=1}^{N} u(w_{t}) - u(w^0_{t}) \right)$ the maximum feasible utility gain, i.e. the utility gain for the principal when the agent invest $\pi$ in every period, chooses the allocation $a^*$ that maximizes the principal’s expected utility, and does not get paid.

**Theorem A.1.** There exists $m > 0$ such that for all $N$ and all $\mathcal{P}$,

$$r_{\lambda,\pi} \geq \frac{1}{\nu} E_{\lambda,\pi} \left( \frac{\Pi_{N}}{N} \right) - \frac{m}{\sqrt{N}} \quad \text{and} \quad r_{\lambda,\pi} \geq \frac{1}{1 + \nu \kappa} r_{\text{max}} - \frac{\bar{c}}{\nu} - \frac{m}{\sqrt{N}}.$$

(35)
Proof. The first inequality follows from (33). Indeed, we have that

\[ S_N + \nu d \sqrt{N} \geq \Pi_N \iff \sum_{t=1}^{N} \nu [u(w^\lambda_t - \pi_t) - u(w^0_t)] \geq \Pi_N - \nu d \sqrt{N} \]

\[ \iff \frac{1}{N} \sum_{t=1}^{N} u(w^\lambda_t - \pi_t) - u(w^0_t) \geq \frac{1}{\nu} \Pi_N - \frac{\nu d}{\sqrt{N}}. \]

The first part of (35) follows directly by taking expectations.

Let us turn to the second inequality. Combining (32) and (33) it follows that

\[ \Pi_N \geq \Sigma_N - \nu d (1 + \sqrt{N}) \geq \sum_{t=1}^{N} \nu [u(w_t - \pi_t) - u(w^0_t)] - \nu d (1 + \sqrt{N}) \]

\[ \geq \sum_{t=1}^{N} \nu [u(w_t) - u(w^0_t) - \kappa \pi_t] - \nu d (1 + \sqrt{N}) \]

\[ \Rightarrow \Pi_N \geq \frac{\nu}{1 + \nu \kappa} \sum_{t=1}^{N} [u(w_t) - u(w^0_t)] - \frac{\nu d}{1 + \nu \kappa} (1 + \sqrt{N}). \]

The agent’s optimal policy \((\tilde{c}, \tilde{a})\) under calibrated contract \((\lambda, \pi)\) must provide the agent with greater utility than \((c^*, a^*)\). Hence

\[ \mathbb{E}_{\tilde{c}, \tilde{a}} \left( \Pi_N - \sum_{t=1}^{N} \tilde{c}_t \right) \geq \mathbb{E}_{\pi, a^*} (\Pi_N - N \bar{c}) \]

\[ \Rightarrow \frac{1}{N} \mathbb{E}_{\tilde{c}, \tilde{a}} \Pi_N \geq \frac{\nu}{1 + \nu \kappa} r_{\text{max}} - \bar{c} - \frac{\nu d}{1 + \nu \kappa} \frac{1 + \sqrt{N}}{N}. \]

This last inequality and the first part of (35) implies the second part of (35). \qed

A.4 Varying Wealth

The calibrated contracts described in Section 4 perform equally well if the invested wealth in each period varies within some set \([0, \bar{w}]\). Let \(w^i_t\) denote the initial invested wealth in period
t. Given a contract \((\lambda, \pi)\), quantities \(\Sigma_T, S_T\) and \(\Pi_T\) are defined as

\[
\Pi_T = \sum_{t=1}^{T} \pi_t; \quad \Sigma_T = \sum_{t=1}^{T} w_t \langle a_t - a^0_t, r_t \rangle; \quad S_T = \sum_{t=1}^{T} \lambda_t w_t \langle a_t - a^0_t, r_t \rangle.
\]

Similarly, let \(\Sigma_{T'} = \Sigma_T - \Sigma_{T'-1}, \Pi_{T'} = \Pi_T - \Pi_{T'-1}, S_{T'} = S_T - S_{T'-1}\). As in Section 4, regrets \(R_{1,T}\) and \(R_{2,T}\) are defined by

\[
R_{1,T} = \Pi_T - \alpha S_T \quad \text{and} \quad R_{2,T} = \max_{T' \leq T} \Sigma_{T' \setminus T'} - S_{T' \setminus T'}.
\]

Contract \((\lambda, \pi)\) is unchanged:

\[
\lambda_{T+1} = \frac{\alpha R_{2,T}^+}{\alpha R_{2,T}^+ + R_{1,T}^+} \quad \text{and} \quad \pi_{T+1} = \begin{cases} 
\alpha \lambda_{T+1} (w_{T+1} - w^0_{T+1})^+ & \text{if } R_{1,T} \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]

Under this adjusted contract, Theorem 2 extends as is, with an identical proof.

### A.5 Perturbed Preferences

A more challenging extension is to allow for the agent’s marginal utility for money to vary over time. In particular, assume that the agent has preferences \(\sum_{t=1}^{N} \mu_t \pi_t\), where \(\mu = (\mu_t)_{t \geq 1}\) is bounded below by \(\mu > 0\). Contract \((\lambda, \pi)\) is the same as in Section 4. Hence, it does not correct for varying marginal utility, which is consistent with the idea that \((\mu_t)_{t \geq 1}\) is an unobserved nuisance parameter. Let \(v = \sum_{t=1}^{N} |\mu_t - \mu_{t+1}|\) denote the total variation of sequence \((\mu_t)_{t \geq 1}\). This section provides an adequate extension of Lemma 2 involving the agent’s perturbed utility. For all \(T \geq 1\), let \(\Pi^\mu_N = \sum_{t=1}^{N} \mu_t \pi_t\), \(\Sigma^\mu_T = \sum_{t=1}^{T} \mu_t (w_t - w^0_t)\) and \(S^\mu_T = \sum_{t=1}^{T} \mu_t \lambda_t (w_t - w^0_t)\). The following result holds.
Lemma A.5 (perturbed incentives).

\[
\sum_{N}^{\mu} - S_{N}^{\mu} \leq \bar{w}(\mu_{N} + v)\sqrt{N} \tag{36}
\]

\[-\alpha w\bar{d}(\mu_{N} + v\sqrt{N}) \leq \Pi_{N}^{\mu} - \alpha S_{N}^{\mu} \leq \alpha w\bar{d}(\mu_{N} + v)\sqrt{N}. \tag{37}
\]

Proof. Let us begin with inequality (36). Using summation by part and Lemma 2, we have that

\[
S_{N}^{\mu} = \sum_{t=1}^{T} \mu_{t}\lambda_{t}(w_{t} - w^{0}_{t}) = \mu_{N}S_{N} + \sum_{t=1}^{N-1} (\mu_{t} - \mu_{t+1})S_{t}
\geq \mu_{N}\Sigma_{N} + \sum_{t=1}^{N} (\mu_{t} - \mu_{t+1})\Sigma_{t} - w\bar{d}\sqrt{N} \left( \mu_{N} + \sum_{t=1}^{N-1} |\mu_{t} - \mu_{t+1}| \right).
\geq \sum_{N}^{\mu} - w\bar{d}(\mu_{N} + v)\sqrt{N}.
\]

Let us turn to (37). Again, Lemma 2 and summation by part implies that

\[
\sum_{t=1}^{N} \mu_{t}\pi_{t} = \mu_{N}\Pi_{N} + \sum_{t=1}^{N-1} (\mu_{t} - \mu_{t+1})\Pi_{t}
\leq \alpha \mu_{N}S_{N} + \alpha \sum_{t=1}^{N-1} (\mu_{t} - \mu_{t+1})S_{t} + \alpha w\bar{d}\sqrt{N} \left( \mu_{N} + \sum_{t=1}^{N-1} |\mu_{t} - \mu_{t+1}| \right).
\leq \alpha S_{N}^{\mu} + \alpha w\bar{d}(\mu_{N} + v)\sqrt{N}.
\]

Finally, a similar argument implies that

\[
\sum_{t=1}^{N} \mu_{t}\lambda_{t}\pi_{t} \geq \alpha S_{N}^{\mu} - \alpha w\bar{d}(\mu_{N} + v\sqrt{N}).
\]

This concludes the proof. \qed

Lemma A.5 implies that under the original calibrated contracts of Section 4, even though the agent’s payoff are perturbed, they still approximate the aggregate payoffs the agent would have received under the benchmark linear contract. Lemma A.5 can be used to derive
efficiency bounds.

**Theorem A.2.** Let $\mu = \mu + v$. There exists $m > 0$ such that for all $N$ and all $\mathcal{P}$,

$$r_{\lambda, \pi} \geq \left(1 - \frac{\alpha}{\alpha \mu} \right) \mathbb{E}_{\lambda, \pi} \left( \frac{\Pi^\mu_N}{\mu N} \right) - \frac{mv}{\sqrt{N}}$$

and

$$r_{\lambda, \pi} \geq \left(1 - \frac{\alpha}{\mu} \left[ \mu r_{\max} - \frac{v}{\alpha w} \right] \right) - \frac{mv}{\sqrt{N}}.$$  \hspace{1cm} (38)

**Proof.** Let us begin with the first part of (38). Let $(\tilde{c}, \tilde{a})$ denote the agent’s optimal policy under the calibrated contract $(\lambda, \pi)$. As in previous sections, $a^*$ denotes the allocation policy that maximizes expected wealth given information. By (37), and by definition of $a^*$ we have that

$$E_{\tilde{c}, \tilde{a}} \Pi^\mu_N - \alpha w d(\mu_N + v) \sqrt{N} \leq \alpha E_{\tilde{c}, \tilde{a}} S^\mu_N \leq \alpha E_{\tilde{c}, \tilde{a}} S^\mu_N.$$  \hspace{1cm} (39)

By definition of $\tilde{a}$ and (37), we also have that

$$E_{\tilde{c}, \tilde{a}} \Pi^\mu_N \geq E_{\tilde{c}, \tilde{a}} S^\mu_N \geq \alpha E_{\tilde{c}, \tilde{a}} S^\mu_N - \alpha w d(\mu_N + v) (1 + \sqrt{N})$$

hence

$$E_{\tilde{c}, \tilde{a}} S^\mu_N - E_{\tilde{c}, \tilde{a}} S^\mu_N \leq w d(\mu_N + v) (1 + 2\sqrt{N}).$$

By definition of $a^*$, for all $t$, $E_{\tilde{c}, \tilde{a}} \lambda_t(w_t - w_t^0) \geq E_{\tilde{c}, \tilde{a}} \lambda_t(w_t^0 - w_t^0)$. Therefore,

$$E_{\tilde{c}, \tilde{a}} S^\mu_N - E_{\tilde{c}, \tilde{a}} S^\mu_N = \sum_{t=1}^{N} E_{\tilde{c}, \tilde{a}} \lambda_t(w_t^0 - w_t^0) - E_{\tilde{c}, \tilde{a}} \lambda_t(w_t - w_t^0)$$

$$\leq \frac{1}{\mu} \sum_{t=1}^{N} \mu_t \left[ E_{\tilde{c}, \tilde{a}} \lambda_t(w_t - w_t^0) - E_{\tilde{c}, \tilde{a}} \lambda_t(w_t - w_t^0) \right] \leq \frac{1}{\mu} \left[ E_{\tilde{c}, \tilde{a}} S^\mu_N - E_{\tilde{c}, \tilde{a}} S^\mu_N \right]$$

$$\leq \frac{1}{\mu} w d(\mu_N + v) (1 + 2\sqrt{N}).$$

This implies that

$$E_{\tilde{c}, \tilde{a}} S^\mu_N + \frac{1}{\mu} w d(\mu_N + v) (1 + 2\sqrt{N}) \geq E_{\tilde{c}, \tilde{a}} S^\mu_N \geq \frac{1}{\mu + v} E_{\tilde{c}, \tilde{a}} S^\mu_N$$

$$\geq \frac{1}{\alpha (\mu + v)} \left[ E_{\tilde{c}, \tilde{a}} \Pi^\mu_N - \alpha w d(\mu_N + v) \sqrt{N} \right].$$
Since (19) still holds, this yields the first part of (38). Let us turn to the second part of (38). By definition of \((\tilde{c}, \tilde{a})\), it must be that

\[
\mathbb{E}_{\tilde{c}, \tilde{a}} \left( \Pi_N^\mu - \sum_{t=1}^N \tilde{c}_t \right) \geq \mathbb{E}_{\tilde{c}, a^*} \left( \Pi_N^\mu - N\tilde{c} \right) \geq \alpha \mathbb{E}_{\tilde{c}, a^*} \sum_N^\mu N - N\tilde{c} - \alpha \widetilde{w}d(\mu_N + v)(1 + 2\sqrt{N})
\]

\[
\geq \alpha \mu \mathbb{E}_{\tilde{c}, a^*} \sum N - N\tilde{c} - \alpha \widetilde{w}d(\mu_N + v)(1 + 2\sqrt{N}).
\]

This and the first part of (38) yields the second part of (38).

B  Proofs

B.1  Proofs for Section 3

Proof of Theorem 1: Points (i) and (ii) follow immediately from the fact that

\[
\pi_t = \alpha (w_t - w_t^0) = \alpha \frac{w_t - w_t^0 - \pi_t}{1 - \alpha}.
\]

Let us turn to point (iii). Let \((c, a^*)\) denote the agent’s policy under the benchmark contract. For any \(c \in [0, \overline{c}]\), denote by \((c, a^*)\) a surplus maximizing policy with expected cost \(c\). Since policy \((c, a^*)\) guarantees the agent a per-period payoff of \(\alpha wr_{\text{max}}(c) - c\), it must be that \(\frac{\alpha}{1 - \alpha} wr_{\alpha} - \mathbb{E}c_{\alpha} \geq \alpha wr_{\text{max}}(c) - c\). Since the agent must expend weakly positive effort, this implies that \(\frac{\alpha}{1 - \alpha} wr_{\alpha} \geq \alpha wr_{\text{max}}(c) - c\), which yields point (iii).

Point (iv) follows immediately from the static nature of the benchmark contract.

Proof of Lemma 1: The fact that benchmark contracts satisfy no-loss follows from point (i) of Theorem 1. Let us turn to the converse.

Contract \((\pi_t)_{t \geq 1}\) induces indirect vNM preferences for the agent and the principal over lotteries with outcomes \((w_t, w_t^0)_{t \geq 1}\). Given such a lottery \(L\), the principal and the agent
respectively have expected utility

\[ \mathbb{E}_L \left( \sum_{t=1}^{N} w_t - w_0^t - \pi_t \right) \quad \text{and} \quad \mathbb{E}_L \left( \sum_{t=1}^{N} \pi_t \right). \]

Because no-loss must hold for every underlying environment \( \mathcal{P} \) and every strategy of the agent, it implies that for every probability distribution \( L \) over outcomes \((w_t, w_0^t)_{t \geq 1}\),

\[ \mathbb{E}_L \left( \sum_{t=1}^{N} w_t - w_0^t - \pi_t \right) \geq 0 \iff \mathbb{E}_L \left( \sum_{t=1}^{N} \pi_t \right) \geq 0. \]

Hence, if \( \mathbb{E}_L (\sum_{t=1}^{N} w_t - w_0^t) = 0 \), then \( \mathbb{E}_L (\sum_{t=1}^{N} \pi_t) \) and \( -\mathbb{E}_L (\sum_{t=1}^{N} \pi_t) \) must have the same sign, which implies that

\[ \mathbb{E}_L \left( \sum_{t=1}^{N} w_t - w_0^t \right) = 0 \implies \mathbb{E}_L \left( \sum_{t=1}^{N} \pi_t \right) = 0. \]

Consider the deterministic sequence such that for all \( t > 1 \), \( w_t = w_0^t = 0 \), \( w_1 = 0 \) and \( w_0^1 = 1 \). Let \( \alpha = -\sum_{t=1}^{N} \pi_t \) for this deterministic sequence of outcomes. Let \( L_{-1} \) denote the lottery putting unit mass on this outcome. For any lottery \( L \) such that \( \mathbb{E}_L (\sum_{t \geq 1} \pi_t) \geq 0 \), consider the compound lottery \( \hat{L} = pL_{-1} + (1 - p)L \), with \( p/(1 - p) = \mathbb{E}_L (\sum_{t \geq 1} \pi_t) \). By construction, \( \mathbb{E}_L (\sum_{t \geq 1} w_t - w_0^t) = 0 \) so that necessarily,

\[ \mathbb{E}_L \left( \sum_{t=1}^{N} \pi_t \right) = 0 \iff -p\alpha + (1 - p)\mathbb{E}_L \left( \sum_{t=1}^{N} \pi_t \right) = 0 \iff \mathbb{E}_L \left( \sum_{t=1}^{N} \pi_t \right) = \alpha \mathbb{E}_L \left( \sum_{t=1}^{N} (w_t - w_0^t) \right). \]

Altogether, this implies that necessarily, for all \( t \), \( \pi_t = \alpha (w_t - w_0^t) \). Finally it is immediate that in order to satisfy no-loss, it must be that \( \alpha \in (0, 1) \).
B.2 Proofs for Section 4

**Proof of Lemma 3:** Under any benchmark linear contract, the agent is truthful – i.e. uses allocation policy $a^*$. Let $(c, a^*)$ denote the agent’s policy under the benchmark contract of parameter $\alpha$, $(\tilde{c}, \tilde{a})$ his policy under contract $(\lambda, \pi)$, and $(c_0, a^*)$ the agent’s policy in the benchmark contract of parameter $\alpha_0$.

By optimality of $(\tilde{c}, \tilde{a})$ under contract $(\lambda, \pi)$, we have that
\[
\mathbb{E}_{\tilde{c}, \tilde{a}} \left[ \Pi_N - \sum_{t=1}^N \tilde{c}_t \right] \geq \mathbb{E}_{c, a^*} \left[ \Pi_N - \sum_{t=1}^N c_t \right].
\]

We obtain that
\[
\mathbb{E}_{\tilde{c}, \tilde{a}} \left[ \alpha \Sigma_N - \sum_{t=1}^N \tilde{c}_t \right] + C \geq \mathbb{E}_{c, a^*} \left[ \alpha \Sigma_N - \sum_{t=1}^N c_t \right] - B \geq \mathbb{E}_{c, a^*} \left[ \alpha \Sigma_N - \sum_{t=1}^N c_t \right] - B - \alpha A. \tag{39}
\]

By optimality of $(c, a^*)$ under the benchmark contract of parameter $\alpha$, we have that
\[
\mathbb{E}_{c, a^*} \left[ \alpha \Sigma_N - \sum_{t=1}^N c_t \right] \geq \mathbb{E}_{c_0, a^*} \left[ \alpha \Sigma_N - \sum_{t=1}^N c_{0,t} \right]. \tag{40}
\]

By optimality of $(c_0, a^*)$ under the benchmark contract of parameter $\alpha_0$ we obtain
\[
\mathbb{E}_{c_0, a^*} \left[ \alpha_0 \Sigma_N - \sum_{t=1}^N c_{0,t} \right] \geq \mathbb{E}_{\tilde{c}, \tilde{a}} \left[ \alpha_0 \Sigma_N - \sum_{t=1}^N \tilde{c}_t \right].
\]

Note that by definition of $a^*$ and $S_N$, $\mathbb{E}_{\tilde{c}, \tilde{a}} \Sigma_N \geq \mathbb{E}_{\tilde{c}, \tilde{a}} S_N$. Indeed, under $a^*$, $\Sigma_T$ delivers positive expected returns every period, while $S_T$ (under any allocation policy) provides at best a fraction of these returns. This implies that
\[
\mathbb{E}_{c_0, a^*} \left[ \alpha_0 \Sigma_N - \sum_{t=1}^N c_{0,t} \right] \geq \mathbb{E}_{\tilde{c}, \tilde{a}} \left[ \alpha_0 S_N - \sum_{t=1}^N \tilde{c}_t \right]. \tag{41}
\]
Combining (39), (40) and (41) yields

\[ \mathbb{E}_{\tilde{c}, \tilde{a}} \left[ \alpha S_N - \sum_{t=1}^{N} \tilde{c}_t \right] + \alpha A + B + C \geq \mathbb{E}_{c_0, a^*} \left[ \alpha \Sigma_N - \sum_{t=1}^{N} c_{0,t} \right] \]

\[ \geq (\alpha - \alpha_0) \mathbb{E}_{c_0, a^*} \Sigma_N + \mathbb{E}_{c_0, a^*} \left[ \alpha_0 \Sigma_N - \sum_{t=1}^{N} c_{0,t} \right] \]

\[ \geq (\alpha - \alpha_0) \mathbb{E}_{c_0, a^*} \Sigma_N + \mathbb{E}_{\tilde{c}, \tilde{a}} \left[ \alpha_0 S_N - \sum_{t=1}^{N} \tilde{c}_t \right]. \]

Altogether, this implies that \((\alpha - \alpha_0) \left[ \mathbb{E}_{c_0, a^*} \Sigma_N - \mathbb{E}_{\tilde{c}, \tilde{a}} S_N \right] \leq \alpha w d (2\sqrt{N} + 1).\) Hence we obtain that

\[ \mathbb{E}_{\tilde{c}, \tilde{a}} [S_N - \Pi_N] \geq (1 - \alpha) \mathbb{E}_{c_0, a^*} \Sigma_N - (1 - \alpha) \frac{\alpha A + B + C}{\alpha - \alpha_0} - C. \]

Dividing by \(N w\), this yields that

\[ r_{\lambda, \pi} \geq (1 - \eta) r_{\alpha_0} - \frac{1}{N w} \left[ C + \frac{1 - \eta}{\eta} (\alpha A + B + C) \right]. \]

\[ \square \]

**Proof of Theorem 3:** Let \(w_t^{\Delta}\) and \(\Sigma_N^{\Delta} = \sum_{t=1}^{N} w_t^{\Delta} - w_t^0\) denote potential realized wealth and aggregate excess returns when accidents are lucky. The notation of Section 4 extends, adding superscripts \(^{\Delta}\) and \(^{\Delta\Delta}\) to denote relevant objects under the original accidental allocation \(a^{\Delta}\), and under the lucky accidental allocation \(a^{\Delta\Delta}\). The key step is to provide an adequate extension of Lemma 2.

Inequality (19) still applies, and we necessarily have that

\[ -\alpha w d \leq \Pi_N^{\Delta} - \alpha S_N^{\Delta} \leq \alpha w d \sqrt{N}. \] (42)
In turn let us show that for any investment strategy of the agent,

$$\Sigma_N^{\Delta_\Delta} - 4w\bar{d}\sqrt{N} \leq S_N^\Delta$$

(43)

i.e. up to an order $\sqrt{N}$, given any investment strategy, the actual excess returns generated under calibrated contract are at least as high as the returns generated when accidents are lucky. We have that $\Sigma_N^{\Delta_\Delta} = \Sigma_{N\setminus T_2+1}^{\Delta_\Delta} + \Sigma_{T_2\setminus T_1}^{\Delta_\Delta} + \Sigma_{T_1-1}^{\Delta_\Delta}$. Because inequality (18) still holds, this implies that

$$\Sigma_N^{\Delta_\Delta} \leq \begin{cases} S_N^\Delta + w\bar{d}\sqrt{N} & \text{if } \Sigma_{T_2\setminus T_1}^{\Delta_\Delta} > 0 \\ S_{N\setminus T_2+1}^\Delta + S_{T_1-1}^\Delta + 3w\bar{d}\sqrt{N} & \text{otherwise} \end{cases}$$

By (42), it follows that

$$\Pi_{T_2}^\Delta - \alpha w\bar{d}\sqrt{T_2} \leq \alpha S_{T_2}^\Delta \leq \Pi_{T_2}^\Delta + \alpha w\bar{d}$$

$$\Pi_{T_1-1}^\Delta - \alpha w\bar{d}\sqrt{T_1-1} \leq \alpha S_{T_1-1}^\Delta \leq \Pi_{T_1-1}^\Delta + \alpha w\bar{d}.$$

Subtracting these two inequalities yields that,

$$\Pi_{T_2\setminus T_1}^\Delta - \alpha w\bar{d}(1 + \sqrt{T_2}) \leq \alpha(S_{T_2}^\Delta - S_{T_1-1}^\Delta) = \alpha(S_{T_2\setminus T_1}^\Delta).$$

Since flow rewards are weakly positive, $\Pi_{T_2\setminus T_1}^\Delta \geq 0$, which implies that for any realization of returns,

$$\Sigma_N^{\Delta_\Delta} \leq S_{N\setminus T_2+1}^\Delta + S_{T_1\setminus T_1}^\Delta + S_{T_1-1}^\Delta + 4w\bar{d}\sqrt{N}$$

$$\leq S_N^\Delta + 4w\bar{d}\sqrt{N}.$$

This implies (43). Given (42) and (43) Theorem 3 follows from a reasoning identical to that of Lemma 3. $\square$
B.3 Proofs for Section 5

The proofs of Lemma 4 and Theorem 5 require the following extension of the Azuma-Hoeffding inequality.

**Lemma B.1** (an extension of Azuma-Hoeffding). Consider a martingale with increments \( \Delta_t \) such that \( |\Delta_t| \leq \gamma \). Filtration \((F_t)_{t \geq 1}\) corresponds to the information available at the beginning of period \( t \). Let \( \gamma_t \equiv \sup |\Delta_t| F_t \) and \( T_m \equiv \inf \{ T \mid \gamma^2 + \sum_{t=1}^{T} \gamma_t^2 \geq m \} \). The following hold.

(i) \( \forall \kappa > 0, \text{Prob} \left( \sum_{t=1}^{T_m} \Delta_t \geq \kappa \right) \leq \exp \left( - \frac{2 \kappa^2}{m} \right) \)

(ii) \( \forall \kappa > 0, \text{Prob} \left( \max_{T \leq T_m} \sum_{t=1}^{T} \Delta_t \geq \kappa \right) \leq 2 \exp \left( - \frac{2 \kappa^2}{m} \right) \).

**Proof of Lemma B.1:** Let us begin with point (i). By Hoeffding’s Lemma, (see Hoeffding (1963) or Cesa-Bianchi and Lugosi (2006), Lemma 2.2) we have that

\[
\mathbb{E}(\exp(-\lambda \Delta_t)|F_t) \leq \exp\left( \frac{\lambda^2 \gamma^2_t}{8} \right).
\]

By construction \( \sum_{t=1}^{T_m} \gamma_t^2 \leq m \). Hence, using Chernoff’s method, we have that for any \( \lambda > 0 \)

\[
\text{Prob} \left( \sum_{t=1}^{T_m} \Delta_t \geq \kappa \right) \leq \exp(-\lambda \kappa) \mathbb{E} \left( \prod_{t=1}^{T_m} \exp(\lambda \Delta_t) \right)
\]

\[
\leq \exp(-\lambda \kappa) \mathbb{E} \left( \exp(\lambda \Delta_1) \mathbb{E} \left( \exp(\lambda \Delta_2) \cdots \mathbb{E} \left( \exp(\lambda \Delta_{T_m}) | F_{T_m} \right) \cdots | F_2 \right) \right)
\]

\[
\leq \exp(-\lambda \kappa) \mathbb{E} \left( \exp \left( \frac{\lambda^2}{8} \sum_{t=1}^{T_m} \gamma_t^2 \right) \right) \leq \exp(-\lambda \kappa) \exp \left( \frac{\lambda^2 m}{8} \right).
\]

Minimizing over \( \lambda \) (i.e. setting \( \lambda = 4\kappa/m \)) yields point (i).

Point (ii) follows from point (i) by adapting the standard reflection techniques used for Brownian motions. Let \( B_T = \sum_{t=1}^{T} \Delta_t \). Pick \( \kappa > 0 \). We want to evaluate \( \text{Prob}(\max_{T \leq T_m} B_T \geq \kappa) \). Consider the process \( \tilde{B}_T = \sum_{t=1}^{T} \epsilon_t \Delta_t \), where \( \epsilon_t = 1_{\max_{s < t} B_s < \kappa} - 1_{\max_{s < t} B_s \geq \kappa} \). Process
\( \tilde{B}_T \) is a martingale, corresponding to reflecting \( B_T \) the first time it crosses level \( \kappa \). Note also that \( |\epsilon_t \Delta_t| = |\Delta_t| \). We have that

\[
\text{Prob} \left( \max_{T \leq T_m} B_T \geq \kappa \right) = \text{Prob}(B_{T_m} \geq \kappa) + \text{Prob}(B_{T_m} < \kappa \text{ and } \max_{T \leq T_m} B_T \geq \kappa) \leq \text{Prob}(B_{T_m} \geq \kappa) + \text{Prob}(\tilde{B}_{T_m} \geq \kappa). \tag{44}
\]

Note that (44) is an inequality, rather than an equality as in the case of a Brownian motion, because of the discreteness of martingale increments. Still this suffices for our purpose. Indeed, by applying point (i) to both \( B_{T_m} \) and \( \tilde{B}_{T_m} \), we obtain that indeed,

\[
\text{Prob} \left( \max_{T \leq T_m} \sum_{t=1}^T \Delta_t \geq \kappa \right) \leq 2 \exp \left( -2 \frac{\kappa^2}{m} \right). \tag{45}
\]

This concludes the proof.

Proof of Lemma 4: We have that

\[
S_T = \sum_{t=1}^T \lambda_t \mathbb{E}_a \left[ w_t - w_t^0 \mid \mathcal{F}_t^0 \right] + \sum_{t=1}^T \lambda_t (w_t - w_t^0 - \mathbb{E}_a \left[ w_t - w_t^0 \mid \mathcal{F}_t^0 \right]).
\]

Since the agent only has access to public information, by definition of \( w_t^0 \), we have that for all allocation strategies \( a \), \( \mathbb{E}_a \left[ w_t - w_t^0 \mid \mathcal{F}_t^0 \right] \leq 0 \). Define \( \Delta_t \equiv \lambda_t (w_t - w_t^0 - \mathbb{E}_a \left[ w_t - w_t^0 \mid \mathcal{F}_t^0 \right])/\sqrt{\kappa^2 + \ln \chi_T} \). \( \Delta_t \) is a martingale increment such that \( |\Delta_t| \leq 2\lambda_t \epsilon_t \).

Let us define \( \chi_T = d^2 + \sum_{t=1}^T \lambda_t^2 \epsilon_t^2 \). For all \( m \in \mathbb{N} \), let \( T_m \) denote the stopping time \( \inf \{ T \mid \chi_T \geq m \} \). Using Lemma B.1, we obtain that for all \( m \)

\[
\text{Prob} \left( S_{T_m} \geq 2w \sqrt{\chi_{T_m}} \sqrt{M + \ln \chi_{T_m}} \right) \leq \text{Prob} \left( \sum_{t=1}^{T_m} \Delta_t \geq 2 \sqrt{\chi_{T_m}} \sqrt{M + \ln \chi_{T_m}} \right) \leq \exp \left( -2 \ln m + M \right) \leq \exp \left( -2M \frac{1}{m^2} \right).
\]

In addition, conditional on \( S_{T_m} \leq 2w \sqrt{\chi_{T_m}} \sqrt{M + \ln \chi_{T_m}} \), Lemma B.1 implies that the
probability that there exists $T \in [T_m, T_{m+1} - 1]$ such that $S_T \geq \Theta_T$ is less than

$$\text{Prob} \left( \sup_{T \in \{T_m, \ldots, T_{m+1}-1\}} \sum_{t=T_m}^{T} \Delta_t \geq 2\sqrt{M + \ln m} \right) \leq 2 \exp(-2M) \frac{1}{m^2}.$$ 

Hence it follows that

$$E_{\alpha} \left( \sum_{t=1}^{N} 1_{S_t > \Theta_t} \right) \leq 3 \exp(-2M) \sum_{m \in \mathbb{N}} \frac{1}{m^2} \leq \frac{\pi^2}{2} \exp(-2M).$$

This concludes the proof. \hfill \Box

**Proof of Lemma 5:** A proof identical to that of Lemma 2 yields the left-hand side of (21) and the right-hand side of (22). The left-hand side of (22) is proven by induction. Assume it holds at time $T$. If $\alpha S_{T+1} - \alpha \Theta_{T+1} \leq 0$, then the inequality holds trivially. Consider now the case where $\alpha S_{T+1} - \alpha \Theta_{T+1} > 0$. If $\Pi_T \geq \alpha S_T - \alpha \Theta_T$ then we necessarily have $\Pi_{T+1} \geq \alpha S_{T+1} - \alpha \Theta_{T+1} - \alpha w d$ since $\Theta_T$ is increasing in $T$. If instead, $\Pi_T \in [\alpha S_T - \alpha \Theta_T - \alpha w d, \alpha S_T - \alpha \Theta_T]$, then necessarily, $\Pi_T \leq \alpha S_T$, so that $\lambda_{T+1} = 1$ and $\pi_{T+1} = \alpha(w_{T+1} - w_0)^+$. It follows that $\Pi_{T+1} \geq \alpha S_{T+1} - \alpha \Theta_{T+1} - \alpha w d$. \hfill \Box

**Proof of Theorem 5:** Consider the case where the agent is informed. As in the case of Theorem 2, 3, and 4, the proof strategy is to adapt the the bounds of Lemma 2 and the reasoning of Lemma 3. Let $(c, a^*)$ denote the agent’s optimal strategy under the benchmark contract of parameter $\alpha$. To exploit the reasoning of Lemma 3 it is sufficient to prove a bound of the form

$$-B \leq E_{c,a^*} \left[ \Pi_N^\Theta - \alpha S_N \right],$$

where $B$ is a number independent of $N$ and $\mathcal{P}$. By construction, we have that

$$E_{c,a^*} \Pi_N^\Theta \geq E_{c,a^*} \alpha S_N - \alpha w d - \alpha w d E_{c,a^*} \left[ \sum_{T=1}^{N} 1_{S_T < \Theta_T} \right].$$

55
Hence, it is sufficient to show that under \((c, a^*)\), the expected number of periods where the hurdle is not met is bounded above by a constant independent of \(N\).

Let \(\Delta_t = w_t - w_t^0 - \mathbb{E}[w_t - w_t^0 | \mathcal{F}_t]\) and \(\chi_T = \frac{d^2}{2} + \sum_{t=1}^{T} d_t^2\). Note that under strategy \((c, a^*)\), Assumption 2 implies that if \(d_t > 0\), then \(\mathbb{E}_{c,a^*}(w_t - w_t^0 | \mathcal{F}_t) > \xi\). Hence \(\sum_{t=1}^{T} \mathbb{E}_{c,a^*}(w_t - w_t^0 | \mathcal{F}_t) \geq \xi(\chi_T/d^2 - 1)\). By (21), for any \(T\),

\[
\begin{align*}
\text{Prob}_{c,a^*}(S_T < \Theta_T) & \leq \text{Prob}_{c,a^*}(\sum_{t=1}^{T} \mathbb{E}[w_t - w_t^0 | \mathcal{F}_t] + \sum_{t=1}^{T} \Delta_t < \Theta_T + w\sqrt{\chi_T}) \\
& \leq \text{Prob}_{c,a^*} \left( \xi \left[ \frac{\chi_T}{d^2} - 1 \right] + \sum_{t=1}^{T} \Delta_t < \Theta_T + w\sqrt{\chi_T} \right) \\
& \leq \text{Prob}_{c,a^*} \left( \sum_{t=1}^{T} \Delta_t < -\xi \left[ \frac{\chi_T}{d^2} - 1 \right] + \Theta_T + w\sqrt{\chi_T} \right).
\end{align*}
\]

An argument similar to that of Lemma 4 yields that \(\sum_{T=1}^{\infty} \text{Prob} \left( \sum_{t=1}^{T} \Delta_t < -\xi \frac{\chi_T}{d^2} + \Theta_T + w\sqrt{\chi_T} \right)\) is bounded above by some constant. This concludes the proof.

\section*{References}


57


*Journal of finance*, 16, 8–37.
