A Theory of Collateral Requirements for Central Counterparties*

Jessie Jiaxu Wang  Agostino Capponi  Hongzhong Zhang
Arizona State University  Columbia University  Columbia University

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Abstract

This paper develops a new framework for the design of collateral requirements in a centrally cleared market. Clearing members post collateral—initial margins and default funds—to increase their pledgeable income, thereby credibly committing to risk management. We show that initial margins are more cost-effective in aligning members’ incentives for risk management. In contrast, default funds are more valuable to enhance the central counterparty’s (CCP) resilience by allowing members to mutualize losses. In setting collateral requirements, the CCP balances the opportunity cost of collateral, its effectiveness in providing incentives, and the cost of CCP recapitalization. Our model predicts extensive use of initial margin during normal times, and of default funds under extreme market scenarios.

Keywords: Central counterparty (CCP), collateral requirements, initial margin, default fund, macro-prudential regulation

JEL: G18, G23, G28, D82, D62

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1 Introduction

A key reform after the financial crisis of 2007-09 is the mandatory clearing of standardized over-the-counter derivatives. The central counterparty (CCP), also known as the clearing-house, enters a trade both as the buyer to the original seller and as the seller to the original buyer. This way, the original counterparties of the trade become insulated from each other’s default risk—provided that the CCP meets its own obligation. With the increasing role of the CCPs,\(^1\) their risk management has drawn the attention of policymakers and the public alike (Cunliffe, 2018; Duffie, 2018, 2019). To ensure its resilience, the CCP collects two types of prepaid collateral from its members: initial margin and default funds. Initial margins are only used to absorb losses triggered by the default of the posting member. Default funds, instead, are shared across members and thus allow for loss mutualization (Pirrong, 2011).

Despite extensive regulatory debates on the clearing reforms, there is little academic research on the structure of the CCP’s default waterfall, and in particular on the design and regulation of margins and default fund requirements.

To the best of our knowledge, this paper provides the first framework to analyze the joint determination of collateral allocated as initial margin and default funds. While posting collateral increases members’ pledgeable income and gives them incentives for risk management, we show that the two types of collateral are imperfect substitutes. Initial margin is more cost-effective in aligning members’ incentives. In comparison, default funds are shared across members to mutualize losses ex-post; hence, they are less effective for ex-ante incentive provision, but more valuable to enhance CCP’s resilience.

Our model features a finite number of risk-neutral dealers, a continuum of risk-averse protection buyers, and the CCP. To seek insurance against credit risk, protection buyers purchase contracts that resemble credit default swaps (CDS) from dealers. A dealer fully invests the payment collected from the buyer in a risky asset. Because of limited liability, a

\(^1\)The volume of cleared contracts has been growing steadily. The global clearing rates for interest-rate and credit derivatives rose from 55% in 2010 to 75% in 2017 for interest rate derivatives, and over the same period they increased from 10% to 55% for credit default swaps (see Bank for International Settlements, 2019).
dealer defaults if a bad state realizes for her asset, and the state realizations are independent across dealers. To partially hedge the asset risk, the dealer can engage in risk management, at the cost of a lower return in the good state. However, the risk-management choice is unobservable. This self-interested unobservable action by dealers, combined with their limited liability, generates moral hazard in terms of insufficient risk management, and we refer to it as risk-shifting.

In a centrally cleared market, the CCP guarantees that all obligations of its clearing members are fulfilled. To achieve this outcome, members post two types of collateral—initial margin and default fund. If a member defaults, the loss that exceeds the member’s total collateral is mutualized among surviving members using the shared default funds. If all collateral resources are exhausted, the CCP can be recapitalized at a cost. Because the CCP insulates each member against counterparty risk, the risk-averse CDS buyers are willing to pay a premium for transacting a centrally cleared CDS. This premium incentivizes dealers to become members of the CCP and participate in central clearing, despite the cost of posting collateral. The CCP, acting as the social planner, sets the optimal collateral requirements to maximize the value of all market participants, subject to the members’ incentive-compatibility constraint.

Our model highlights the incentive role of collateral. In the first-best benchmark, all members engage in risk management, thus costly collateral of neither type is needed to provide incentives. By contrast, when members’ risk-management choices are unobservable, moral hazard occurs, and members’ incentives for risk-shifting dampen as collateral increases. Without collateral, members prefer not to hedge their risks. Posting a sufficient amount of collateral increases members’ pledgeable income (Holmstrom and Tirole, 1997), allowing them to commit to efficient risk management credibly.

The two types of collateral, however, are not perfect substitutes; they affect CCP’s resilience and members’ risk-shifting incentives differently. We show that an additional unit of collateral posted as initial margin is more effective in aligning members’ incentives than if posted as default fund. Intuitively, initial margin contributes exclusively to the posting
member’s pledgeable income. Default fund, on the other hand, is shared across members to mutualize losses, thereby contributing less to the pledgeable income of the posting member. Although this feature makes default fund more valuable to CCP’s resilience, it undermines the effectiveness of incentive provision.

The optimal incentive-compatible collateral arrangements depend on the opportunity cost of posting collateral, its effectiveness in providing incentives, and the cost of CCP recapitalization. The CCP trades off between minimizing members’ opportunity cost of posting collateral and reducing the expected recapitalization cost. If collateral opportunity costs are higher (such as in tranquil times or during economic expansions), the optimal collateral policy relies more on initial margin, which is a more cost-effective tool to provide incentives. In contrast, if concerns about the CCP’s resilience dominates (e.g., during periods of market distress when recapitalizing the CCP can be very costly), demanding more default fund contributions from members to mutualize losses is socially preferable.

Our finding offers a rationale for collecting default funds during periods of market stress when the cost of CCP recapitalization is high. Moreover, the default fund level predicted by our model differs from the current CPSS-IOSCO (2012) international regulatory guideline known as the “Cover 2” rule. We show that, for clearinghouses consisting of many members, the total default funds collected should cover the shortfalls of a certain fraction of clearing members and respond to entry and exit of members in the clearing business. This finding is in line with Murphy and Nahai-Williamson (2014) who argue that the “Cover 2” standard is far from prudent.

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2Under the prevailing rule, known as the “Cover 2,” the total default fund posted by members should be sufficient to cover the losses caused by the defaults of the two largest clearing members. The Committee on Payment and Settlement Systems (CPSS) and the International Organization of Securities Commissions (IOSCO) issued principles that require CCPs to maintain a default fund large enough to cover the default of two members in extreme yet plausible market scenarios. Clearinghouses in the US must abide by a “Cover 1” system at a minimum, while international and systemic US clearinghouses must comply with the CPSS-IOSCO (2012) regulatory guidelines. Major derivative clearinghouses, such as ICE Clear Credit, CME Clearing, ICE Clear, and LCH.Clearnet, adopt the “Cover 2” rule (Arnakola and Laurent, 2017).

3Murphy and Nahai-Williamson (2014) [pg. 17] argue that higher levels of financial resources may be needed to ensure the robustness of the clearinghouse: “Perhaps a simple backstop to cover 2 could be considered, such as demanding that the default fund in addition meets the requirement that it is larger than some fixed percentage of the ‘cover all’ requirement.”
We examine the robustness of our baseline model by incorporating two realistic features of the clearing business. First, when multiple defaults occur and all prepaid collateral resources are exhausted, the market is likely distressed, and access to end-of-waterfall resources can become increasingly costly. To account for this scenario, we consider a variant of the model where the CCP faces convex, as opposed to linear, recapitalization costs. We show that the fundamental economic trade-off from the baseline model continues to hold. Moreover, the nonlinearity allows us to pin down collateral allocations that consist of both initial margin and default fund. Second, we allow for size heterogeneity in members' outstanding positions, capturing the fact that CCPs' outstanding exposures are concentrated in a few large clearing members (e.g., Office of Financial Research, 2017). Our main results are qualitatively robust to size heterogeneity, and the optimal collateral rule becomes size-dependent. Unlike the banking literature, which has shown that big banks tend to take excessive risk (e.g., O’Hara and Shaw, 1990; Nosal and Ordonez, 2016; Davila and Walther, 2019), we find that it is easier for big members to internalize the externalities from loss mutualization, whereas small members have stronger incentives for risk-shifting. Consequently, the required collateral is disproportionately lower for the big members.

Our paper adds to existing literature on the role of collateral in derivative contracts. Oehmke (2014) develops a model of collateral liquidation, and shows that margin setting by repo lenders should reflect their balance sheet constraints and strategic interactions during liquidation. Bolton and Oehmke (2015) demonstrate that, while margins provide incentives for counterparties, seniority for derivatives transfers risk to creditors, thereby increasing the cost of debt. Biais et al. (2016) show that margin calls have an incentive role after the arrival of bad news that increases protection sellers’ expected liabilities. Biais et al. (2019) further examine the constrained-efficiency of variation margins when margin calls trigger fire sales. The novelty of our analysis is on the composition of collateral, by modeling the economic differences between initial margin and default funds in the context of central clearing.

\footnote{We refer to Anderson and Jöeveer (2014) for a discussion of initial margin rules in the context of central clearing, and to Duffie et al. (2015), Capponi et al. (2017), and Cruz Lopez et al. (2017) for empirical analyses of initial margin requirements of CDS clearinghouses.}
The loss-mutualization through default funds is related to insurance (Arrow, 1974; Raviv, 1979; Parlour and Plantin, 2008). Like the insurance industry, risk-shifting arises when risk is not entirely borne by the agent. Nevertheless, central clearing presents unique distinguishing features from insurance. First, unlike the insurance industry where the moral hazard problem is for the insured party, in central clearing moral hazard concerns pertain to clearing members who sell insurance. Second, default funds are collected as collateral and differ from insurance guaranty funds which operate on a post-assessment basis. A higher post-assessment increases risk-taking (Lee et al., 1997) and generates the opposite incentive than in central clearing.

Our paper contributes to the debate on CCP regulation. While CCPs are expected to improve financial stability, a growing literature has highlighted various forms of inefficiency that potentially arises under central clearing (see, for example, Duffie and Zhu, 2011). Other studies have quantitatively assessed the resilience of CCPs using stress testing methods, and called for better design of the CCP default waterfall mechanism (Boissel et al., 2017; Menkveld, 2017; Paddrik et al., 2019). Legal scholars have also echoed concerns on the current design of CCPs (see, for instance, Yadav (2013) and Roe (2013)). Our paper contributes to this policy debate by providing a normative analysis on the design of collateral requirements, which accounts for the two critical policy considerations: clearing members’ risk-management incentives and the resilience of the CCP.

The paper proceeds as follows. Section 2 sets up the model, Section 3 studies the optimal

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5After an insurance provider becomes insolvent, the solvent companies are assessed an amount equal to the shortfall at the insolvent firm (Cummins, 1988).

6Theoretical studies have analyzed the potential benefits of central clearing in increasing transparency (Acharya and Bisin, 2014), economizing on settlement costs (Koeppel et al., 2012), achieving loss mutualization (Zawadowski, 2013), and pooling counterparty risks (Stephens and Thompson, 2014). A few empirical studies also document the value of CCPs in reducing counterparty risk (Loon and Zhong, 2014; Bernstein et al., 2019), enhancing price stability (Menkveld et al., 2015), completing markets (Vuillemey, 2019b), and mitigating fire sales (Vuillemey, 2019a).

7Duffie and Zhu (2011) show that central clearing could increase counterparty risk if the clearing process is fragmented across multiple CCPs; relatedly, Duffie et al. (2015) use CDS data to show that central clearing does not necessarily reduce collateral demand; Huang (2019) shows that a for-profit CCP may have incentives misaligned with financial stability, in that it contributes less equity and charges lower collateral than socially optimal; Pirrong (2014) and Spatt (2017) argue that central clearing reforms may redistribute, rather than reduce, risk; and Blais et al. (2012) and Kubitza et al. (2019) show that central clearing is not effective to insure aggregate risk.
design of collateral requirements. Section 4 investigates the robustness of our model by incorporating more realistic features of the clearing business. Section 5 discusses the policy and empirical implications, and Section 6 concludes. Proofs are delegated to the Appendix.

2 Model

We develop a parsimonious model to study how different types of collateral (i.e., initial margins and default funds) affect members’ risk-taking incentives and the CCP’s resilience. Our setup is related to that of Biais et al. (2016). There are two dates, \( t = 0, 1 \), a continuum of risk-averse protection buyers, \( N \) risk-neutral dealers who sell swap contracts that resemble CDSs, and a CCP. At \( t = 0 \), the parties design and enter the contract; risk-management decisions are made. At \( t = 1 \), payoffs are realized.

Agents and Assets. Protection buyers are identical, with mean-variance preferences and risk-aversion parameter \( \gamma > 0 \). They are endowed with one unit of an asset whose random return is realized at \( t = 1 \): The return is \(-D\) if an aggregate credit event occurs, and zero otherwise. The credit event happens with probability \( p_c \in (0, 1) \).

Protection buyers seek insurance against credit risk from dealers, who are risk-neutral and have limited liability. According to the insurance contract, a buyer pays an amount \( A \) to a dealer at \( t = 0 \); in exchange, the dealer promises to pay \( D \) to the buyer at \( t = 1 \) upon the occurrence of the credit event. The contract resembles a CDS: It swaps an upfront payment today for a promise to receive a random amount at \( t = 1 \). We use the institutional features of a CDS for concreteness, but our model is general enough to accommodate any class of contracts in which the counterparty risk is one-sided.8 Following Biais et al. (2016), we model the credit event as being systematic, so that our contracts effectively resemble index CDSs, rather than single-name CDSs. Such a modeling choice is strongly supported by empirical evidence, as index contracts account for more than 99% of centrally cleared CDS contracts.

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8In our model, buyers cannot self-insure or trade alternative contracts that deliver a future payment upon the credit event. This assumption guarantees that buyers value CDS contracts, hence granting an economically meaningful role to these contracts.

Each dealer sells one unit of CDS and fully invests the payment collected from the buyer, \( A \), in a risky and productive asset.\(^9\) Dealers have unique skills to manage the risky asset and extract excess return; protection buyers and the CCP are not endowed with those skills. The risky asset generates a per-unit return of zero in the bad state, which occurs with probability \( q_r \), and of \( R_r \) in the good state which occurs with probability \( 1 - q_r \). The return of the risky asset is assumed to be independent across dealers, and also independent of the credit event.\(^10\)

At \( t = 0 \), each dealer makes a risk-management decision to partially hedge its asset risk. To model risk management in a parsimonious fashion, we assume that each dealer can choose whether or not to make her asset safer, i.e., to reduce the probability of the bad state realization to \( q_s \), at the cost of a lower return \( R_s \) in the good state, i.e.,

\[
0 < q_s < q_r, \quad R_s < R_r. \tag{1}
\]

The risk-management process reflects the unique skills of the dealer and is therefore difficult for outside parties to observe and monitor. Combined with limited liability of the dealer, the upside potential of the risky asset generates moral hazard in terms of insufficient hedging.\(^11\) Denote the risk-management choice by \( a \in \{ s \) (to hedge), \( r \) (not to hedge)\}. The expected return of the asset is thus \( (1 - q_a)R_a \), which we denote as \( \mu_a \). We assume the expected return is higher when hedging is conducted than otherwise, so risk management is efficient, i.e.,

\[
\mu_s > \mu_r. \tag{2}
\]

We thus refer to the risk-management choice \( a_i = s, \forall i \), as the first-best benchmark, and refer to \( a_i = r \) as dealer \( i \) engaging in risk-shifting.

\(^9\)For analytical tractability, we assume dealers have the same size by normalizing the size of CDS contracts sold to one. Nonetheless, since the investment opportunity is linearly scalable, we consider an extension where sizes are heterogeneous in Section 4.2.

\(^10\)The asset return models the asset side of the balance sheet, which we naturally think of being not directly related to the CDS contract. While one could introduce a systematic risk component in the return outcome of risky assets, the idiosyncratic risk component is the necessary ingredient to make loss-sharing meaningful.

\(^11\)Because of the upside potential of the risky asset, \( R_r > R_s \), our model does not require costly effort to generate moral hazard as in e.g., Zawadowski (2013) and Biais et al. (2016).
Collateral and Central Clearing. Under central clearing, CDS dealers establish direct contractual relationships with the CCP. Acting as a public utility, the CCP does not have access to risky assets and thus has no profit-making incentive. Its role is to fulfill the promised payments to buyers, and to design collateral rules that maximize the value of its members (i.e., it acts as a benevolent social planner).

At $t = 0$, the CCP collects two types of collateral from its members—initial margin and default fund. Both types of collateral are deposited into an escrow account in the form of safe assets (cash or Treasuries), thereby imposing a marginal opportunity cost to the member, equal to $\beta > 0$.

Using the collateral resources, the CCP mutualizes losses caused by the default of members, thereby guaranteeing the promise to CDS buyers at $t = 1$. The mechanism behind the loss-mutualization is the CCP’s default waterfall; see Figure 1. Initial margins, denoted by $I \in [0, D]$, serve as the first line of defense against losses when the posting member cannot fulfill the promise $D$ to the buyer. The default fund, $F \in [0, D - I]$, constitutes the second loss-absorption layer of the default waterfall. Unlike initial margins which are used to cover losses of the posting member only, default funds are shared among members, which leads to the third layer of the default waterfall. Losses exceeding the defaulting member’s collateral are allocated proportionally to the default funds of surviving members. Hence, a surviving member might incur a loss when other members default, creating an externality among members.

The last layer consists of the end-of-waterfall resources. Multiple clearing members could

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12 While the CDS buyers could also be members of the CCP, we focus on the risk-shifting incentives of CDS dealers because their risk-management choice is non-contractible, and they are subject to default. Consequently, the CCP has risk exposure to only the CDS dealers, not to buyers.

13 Clearing members maintain both proprietary positions, for which they post margins directly, and client accounts to clear trades on behalf of their clients. In the latter case, initial margins are posted by clients. The gross notional and number of contracts traded on the members’ proprietary accounts are typically larger than the corresponding quantities for the client positions. Capponi et al. (2017) provide supporting empirical evidence for ICE Clear Credit.

14 Different from Biais et al. (2016), in our model the amount of collateral does not affect the member’s investment, $A$, in the risky asset. The collateral presents an additional opportunity cost, capturing either funding costs or the effects of foregone investment opportunities.

15 There is not yet a universally agreed-upon loss allocation rule. Major clearinghouses, such as the ICE Clear Credit, adopt a pro-rata basis for futures, options, and CDS contracts (ICE, 2016).
default at the same time. If the total collateral resources posted by members are insufficient to absorb losses, the CCP needs to be recapitalized with additional resources. Let $N_d$ be the number of defaulting members, i.e., $N_d = \sum_i 1_{i \text{ defaults}} = \sum_i 1_{\text{credit event } \cap \text{bad state for } i}$, then $(N_d(D - I) - NF)^+$ denotes losses that are not covered by the prefunded collateral resources. The total cost of recapitalizing the CCP is linear in the unfunded shortfall and given by $(1 + \beta)(N_d(D - I) - NF)^+$, where $\beta$ is the marginal opportunity cost of collateral.

Our modeling of the default waterfall structure makes two simplifying assumptions. First, the CCP can always retrieve enough resources (at a cost) to honor the obligations towards buyers in all states of the economy. In other words, the CCP never actually “defaults” in our model. This assumption is supported by the current stringent end-of-waterfall risk-management practices, and the regulatory progress of CCPs’ failure resolution (Cunliffe, 2018). Second, we do not consider the so-called “CCP’s skin-in-the-game.” Although in

![Figure 1. CCP Default Waterfall Structure.](image)

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16 The end-of-waterfall resources include additional contributions from the CCP’s own capital and by non-defaulting clearing members. If these measures are still insufficient to absorb the losses, the CCP restructures its clearing payment obligations to clearing members through variation margin gains haircuttering (CPMI-IOSCO, 2014; Cont, 2015; Duffie, 2015).

17 From Cunliffe (2018): “Unlike banks, if prefunded resources are insufficient to absorb losses, the CCP does not become insolvent. CCP’s rulebooks, to which all members agree in advance, allow a further series of recovery actions, in a prescribed sequence. A CCP could require its members to contribute more resources, it could reduce its liabilities to some of its members, and ultimately it could “tear-up” (i.e. cancel) all or some
practice most CCPs absorb losses exceeding the collateral of a defaulting member using their own equity before using funds of surviving members, the CCP’s equity contribution to the default waterfall is typically small relative to the collateral posted by members.\footnote{The filings of 60 CCPs, in compliance with the Principles for Financial Market Infrastructures (CPMI-IOSCO, 2015), reveal that the equity contribution of an average CCP clearing interest rate swaps is only 1.6\% of its prefunded resources, while the corresponding figure for a CCP clearing credit default swaps is on the order of 2\% (Paddrik and Zhang, 2019).} Moreover, the seniority structure in practice is intended to reduce moral hazard on the part of the CCP (Duffie, 2015; Saguato, 2017; Huang, 2019). In our model and as Cunlíffe (2018) points out, “CCPs are not risk-taking entities. Rather, they are mechanisms for the management and for the mutualization of their members’ counterparty risk. Their greatest, though not only, vector of risk is the default of one or more of their members.”

### Pricing of Bilateral and Centrally Cleared Contracts

The price of the CDS contract is determined by the counterparty risk and buyers’ risk-aversion. We make the following assumption on protection buyers.

**Assumption 1** Protection buyers have zero bargaining power in the pricing of a CDS contract. Moreover, their risk-aversion parameter satisfies $\gamma > \gamma$, where the explicit expression of $\gamma$ is given by Eq. (A12) in the Appendix.

Under Assumption 1, a dealer has all the bargaining power and is able to extract monopoly rent (so protection buyers are held to their reservation utility). When matched with one protection buyer, each dealer makes an exclusive take-it-or-leave-it offer.\footnote{This assumption is supported by the empirical observation that derivatives markets tend to be concentrated on a small number of dealers who can then charge markups from end-users (Brunnermeier et al., 2013; Peltonen et al., 2014; Li and Schurhoff, 2019; Siriwardane, 2019). Consistent with this assumption, the CCP (acting as a social planner) places no weight on protection buyers.} We also assume that the protection buyers are sufficiently risk averse, in that they attach a large enough value to a centrally cleared CDS contract.

Let $A_{BT}$ be the price of a bilaterally traded CDS contract, and $A_{CCP}$ be the price of a centrally cleared CDS contract. A protection buyer in autarky has a payoff of zero in the
absence of the credit event and a payoff of $-D$ otherwise, so her mean-variance utility is

$$-p_cD - \gamma p_c (1 - p_c) D^2. \quad (3)$$

Holding a bilaterally traded CDS contract, a protection buyer reduces her credit risk exposure to $p_c q_a$. She has a payoff of $-A_{BT}$, unless the dealer defaults in a credit event, which occurs with probability $p_c q_a$ and leaves the buyer with a payoff of $-D - A_{BT}$.\(^{20}\) Hence, the utility of the protection buyer becomes

$$-p_c q_a D - \gamma p_c q_a (1 - p_c q_a) D^2 - A_{BT}. \quad (4)$$

Finally, a protection buyer holding a centrally cleared CDS contract has a constant payoff of

$$-A_{CCP}. \quad (5)$$

We solve for $A_{BT}$ and $A_{CCP}$. In a bilateral setting, the buyer is offered a price $A_{BT}$ such that her utility in (4) is at least as large as her reservation value in (3), for any choice $a \in \{s, r\}$. In a centrally cleared setting, instead, the buyer is offered a price $A_{CCP}$ that makes her utility in (5) equal to her reservation value. This yields the prices

$$A_{BT} = p_c D (1 - q_r) (1 + \gamma D (1 - p_c - p_c q_r)); \quad (6)$$

$$A_{CCP} = p_c D (1 + \gamma D (1 - p_c)). \quad (7)$$

When a CCP guarantees the trade, the payment obligation of the CDS contract is honored with certainty, so the buyers are hedged against counterparty risk. Buyers value this guarantee because they are risk averse. Hence, they are willing to pay a positive premium $A_{CCP} - A_{BT} = p_c q_r D + \gamma p_c q_r (1 - p_c q_r) D^2$ for a centrally cleared CDS.\(^{21}\) Our model thus implies that central clearing reduces counterparty risk and increases the CDS price.\(^{22}\) We

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\(^{20}\)The bilaterally traded CDS price ($A_{BT}$) is not contingent on dealer’s risk-management choice ($a$). This is because the risk-management choice is not observable nor contractible, even though the counterparty risk $p_c q_a$ depends on $a$.

\(^{21}\)This premium consists of two parts: the expected loss due to default of the counterparty under the credit event, $p_c q_r D$, and the compensation for the reduction in the variance of payoffs, $\gamma p_c q_r (1 - p_c q_r) D^2$. The higher the risk aversion $\gamma$, the higher the premium earned by dealers.

\(^{22}\)In the context of single-name CDSs, Loon and Zhong (2014) find that the spreads of centrally cleared
further impose a few technical restrictions on the model parameters.

**Assumption 2** The returns of the risky asset satisfies

$$\frac{\mu_s - \mu_r}{q_r - q_s} < \frac{1}{1 + \gamma D(1 - p_c)}; \quad (8)$$

$$p_c R_s (1 - q_r) > \frac{1}{1 + \gamma D(1 - p_c - p_c q_r)} > 0. \quad (9)$$

Condition (8) is equivalent to $D - \frac{A_{CCE}(\mu_s - \mu_r)}{(q_r - q_s)p_c} > 0$. It implies that, without collateral requirements, a dealer prefers not to hedge; see Section 3.1 for more discussion. Condition (9) guarantees that the CDS price $A_{BT}$ is positive and that a dealer who engages in risk management can fully insure buyers in the good state ($A_{BT} R_s > D$).

### 3 Risk-shifting Incentives and the Design of Collateral

In this section, we study members’ incentives in undertaking risk management and the optimal design of collateral requirements. Section 3.1 studies the value of a dealer in a market where contracts are traded bilaterally without any posting of collateral. Sections 3.2 and 3.3 consider the setting in which traded contracts are centrally cleared, focusing respectively on the first-best benchmark where risk-management choices are imposed by a benevolent social planner, and the second-best outcome where members selfishly decide on risk management. Section 3.4 characterizes the optimal collateral requirements under the second-best outcome.

#### 3.1 Dealer’s Value in a Bilateral Market

Recall that under the assumption stated in condition (2), risk management is efficient, i.e., hedging improves the expected return of a dealer. Next we analyze whether private and social incentives are aligned if dealers trade CDS contracts bilaterally without posting collateral to protection buyers.

A dealer who makes the risk-management choice $a \in \{s, r\}$ achieves a payoff from the risky asset equal to $A_{BT} R_a$ in the good state, except if the credit event occurs and the dealer
fully honors her promise $D$ to the buyer. In the bad state, which occurs with probability $q_a$, the dealer receives a zero payoff. Let $V_{BT}$ be the expected profit of a dealer in a bilateral market; $V_{BT}$ is given by

$$V_{BT} = \max_{a \in \{s,r\}} (1 - q_a)(A_{BT}R_a - p_cD) = A_{BT}\mu_r - (1 - q_r)p_cD,$$  \hspace{1cm} (10)$$

where the last equality follows directly from condition (8).\textsuperscript{23} Hence, a CDS dealer chooses not to hedge when she trades bilaterally without posting any collateral. Limited liability undermines a dealer’s incentive to hedge, as she loses the upside potential of the risky asset, whereas the benefit of staying solvent accrues in part to the protection buyers. The mechanism for risk-shifting is reminiscent of the debt overhang problem, first analyzed by Myers (1977).

### 3.2 Central Clearing: the First-Best Benchmark

**Value of a Clearing Member.** Unlike in the bilateral market, a dealer who is a clearing member posts initial margin $I$ and default fund $F$. Posting collateral is costly. Given the collateral requirements and risk-management choices by others, member $i$ makes risk-management decisions to maximize her expected profit at $t = 1$. Let $V(a_i, a_{-i}; I; F)$ denote the expected profit of dealer $i$ in a centrally cleared market. This profit depends on the dealer’s choice $a_i \in \{r, s\}$, the choice of other dealers $a_{-i}$, and the collateral $(I, F)$, i.e.,

$$V(a_i, a_{-i}; I; F) = -(1 + \beta)(I + F) + (1 - p_c)(I + F) + (1 - q_{a_i})(A_{CCP}R_{a_i} - p_cD + p_cI + p_c\mathbb{E}^a\left[\left(F - \frac{N_d(D - I - F)}{N - N_d}\right)^+\right]),$$

where $\mathbb{E}^a[\cdot]$ denotes the expectation conditional on the occurrence of the credit event and members’ risk-management choice $a$.

The first term in $V(a_i, a_{-i}; I; F)$ represents the opportunity cost of collateral. To understand the remaining terms, we list all possible outcomes at $t = 1$. In the good state (with

\textsuperscript{23}Condition (8) and $A_{CCP} - A_{BT} > 0$ jointly imply that $(1 - q_s)(A_{BT}R_s - p_cD) < (1 - q_r)(A_{BT}R_r - p_cD)$, which proves Eq. (10).
probability $1 - q_a$), member $i$ obtains investment proceeds $A_{CCP} R_a$.

(i) With probability $p_c$, the credit event occurs. Member $i$ delivers $D$ to the buyer and retrieves the initial margin collateral $I$. The default fund posted by $i$ is used to absorb the shortfalls of defaulting members, and the residual if any, $\left( F - \frac{N_d(D-I-F)}{N-N_d} \right)^+$, is returned to $i$. This term reflects the loss-mutualization mechanism. Recall that $N_d$ denotes the number of defaulted members when the credit event occurs. The total shortfall beyond the defaulted members’ collateral, $N_d(D-I-F)$, is shared equally by surviving members. Hence, member $i$ incurs a loss of $\frac{N_d(D-I-F)}{N-N_d}$ capped at the default fund.

(ii) If the credit event does not occur, member $i$ gets back all segregated collateral, $I + F$.

If, instead, the bad state occurs (with probability $q_a$), member $i$ obtains zero return from the asset.

(i) With probability $p_c$, the credit event occurs. The member’s initial margin and default fund cover partially the promised payment $D$ to the CDS buyer. Member $i$ achieves a zero payoff.

(ii) If the credit event does not occur, member $i$ gets back the entire amount of collateral.

**First-Best Benchmark.** Having defined the value of clearing members, we consider the case in which dealers’ risk-management choices are observable. The first-best is achieved, and there is no risk-shifting. While implausible, this case offers a benchmark against which we can identify the inefficiencies that arise when dealers’ risk-management choices are not observable.

In the first-best benchmark, the CCP (who acts as the social planner) chooses clearing members’ hedging actions, $a^{FB}$, and collateral requirements, $(I^{FB}, F^{FB})$, to maximize the sum of all clearing members’ expected profits given by Eq. (11), net of the CCP’s expected
contribution to the default waterfall:\footnote{Recall from Assumption 1 that protection buyers have zero bargaining power in the pricing of CDS contracts; they are held to their reservation utility, which is independent of the CCP’s choice variables. For this reason, the CCP places no weight on protection buyers when maximizing the total value of market participants.}

\[
\max_{(a; I; F)} W^{FB}(a; I; F) \equiv \max_{(a; I; F)} \frac{1}{N} \left\{ \sum_{i} V(a_i, a_{-i}; I; F) - (1 + \beta)p_c \mathbb{E}_a \left[ (N_d(D - I) - NF)_{+} \right] \right\}.
\] (12)

Above, \( W^{FB}(a; I; F) \) is 1/N of the objective function in the first-best benchmark, and \( \beta \) is the recapitalization cost faced by the CCP to retrieve end-of-waterfall resources, assumed to be the same as the collateral opportunity cost for members. Proposition 1 characterizes the first-best outcome.

**Proposition 1** The first-best risk-management choice is \( a_i^{FB} = s, \forall i \); the first-best collateral is \( I^{FB} = F^{FB} = 0 \).

In the first-best, all dealers are requested to hedge their asset risk because doing so increases the expected value (see condition (2)). Counterparty risk is kept at a minimum but remains positive and equal to \( p_c q_s \). In case of members’ defaults, the CCP mutualizes the losses and fully guarantees promise \( D \) to protection buyers, incurring a cost of recapitalization. Collateral of neither type is used because it is costly and offers no benefit when the hedging choice is observable. This proposition thus shows that costly collateral is collected only as an incentive device.

### 3.3 Central Clearing: Members’ Risk-shifting Incentives

We consider the case in which dealers’ risk-management choices are not observable. Like in the uncollateralized bilateral market discussed in Section 3.1, clearing members prefer not to hedge if collateral requirements are absent in the centrally cleared market.\footnote{Condition (8) is equivalent to \( V(a_i = s; I = F = 0) < V(a_i = r; I = F = 0), \forall a_{-1}, i.e., a clearing member prefers not to hedge.} Next, we analyze how collateral requirements affect members’ incentives for risk-shifting.

Posting collateral increases the *pledgable income* (see Holmstrom and Tirole (1997)) and allows members to credibly commit to hedging. Consider, for instance, the extreme case of a
fully collateralized position: members would never default, and limited liability is essentially irrelevant. Hence, condition (2) suggests that all members choose to hedge. Nevertheless, a fully collateralized position is too costly. Central clearing, by mutualizing default losses, saves on collateral cost while providing full insurance to buyers. Two key questions arise: How much collateral is sufficient? Are the two types of collateral perfect substitutes?

We show that an additional unit of collateral posted as initial margin is more effective in aligning members’ incentives than if posted as default fund. Intuitively, initial margin contributes exclusively to the posting member’s pledgeable income. Default fund, on the other hand, is shared across members to mutualize losses. Although this feature makes default fund more valuable to CCP’s resilience, it undermines the effectiveness of incentive provision. Hence, the difference between initial margin and default fund in providing incentives is closely tied to the loss-mutualization mechanism, which we now discuss in detail.

Suppose the credit event occurs, a good realization for the risky asset invested by member $i$ occurs, and $g$ of the other members choose to hedge. The expected refund of the segregated default fund, and thus the effective contribution of default fund to the posting member’s pledgeable income, is given by

$$
\psi(g; I; F) = \mathbb{E} \left[ \left( F - \frac{N_d(D - I - F)}{N - N_d} \right)^+ \mid a_{-i} = s = g \right],
$$

(13)

where $|a_{-i} = s|$ denotes the number of other members who choose to hedge.\(^{26}\) We show in Lemma IA.1 of the Internet Appendix that the function $\psi(g; I; F)$ is piecewise linear in both $I$ and $F$, and strictly increasing in all of the three arguments. Hence, posting a higher collateral means a higher refund, and a higher pledgeable income. Note that $\psi(g; I; F) < F$: The effective pledgeable income is less than the default fund posted, a consequence of other members imposing externality on member $i$. This externality undermines the effectiveness

\(^{26}\)The explicit expression is $\psi(g; I; F) = \sum_{N_d=0}^{\lfloor NF \rfloor} f_d(N_d) \left( F - \frac{N_d(D - I - F)}{N - N_d} \right)$, where $\lfloor \cdot \rfloor$ is the floor function. The term $F - \frac{N_d(D - I - F)}{N - N_d}$ is the refund to member $i$ if $N_d$ of the $N - 1$ members default. The term $f_d(N_d)$ is the probability of such an event when $g$ of the remaining members choose to hedge. If the number of defaults exceeds $\lfloor NF \rfloor$, the total collateral is not sufficient to meet the aggregate payment shortfall, so the refund to member $i$ is zero.
of default fund in providing members incentive to hedge. The next proposition summarizes our main results on the risk-management strategy of clearing members.

**Proposition 2** Given collateral requirements \( I \) and \( F \), members’ risk-management decision rules satisfy

\[
a(I; F) = \arg \max_{a \in \{s, r\}} V(a_i, a_{-i}; I; F) = \begin{cases} 
  r, \forall i & 0 \leq F < \hat{F}(I) \\
  r, \forall i, or s, \forall i & \hat{F}(I) \leq F \leq \bar{F}(I) \\
  s, \forall i & \bar{F}(I) < F \leq D - I
\end{cases}
\]  

(14)

The cutoff values \( \hat{F}(I) \) and \( \bar{F}(I) \) are uniquely determined by the following equation

\[
I + \psi(N - 1; I; \hat{F}) = I + \psi(0; I; \bar{F}) = D - \frac{\mathcal{ACC} (\mu_s - \mu_r)}{(q_r - q_s)p_c}.
\]  

(15)

Both \( \hat{F}(I) \) and \( \bar{F}(I) \) are piecewise linear and strictly decreasing functions of \( I \) with slopes less than \(-1\). Moreover, \( \hat{F} \left( D - \frac{\mathcal{ACC} (\mu_s - \mu_r)}{(q_r - q_s)p_c} \right) = \bar{F} \left( D - \frac{\mathcal{ACC} (\mu_s - \mu_r)}{(q_r - q_s)p_c} \right) = 0 \), and \( 0 < \hat{F}(I) < \bar{F}(I) < D - I \) for all \( I \in \left[ 0, D - \frac{\mathcal{ACC} (\mu_s - \mu_r)}{(q_r - q_s)p_c} \right] \).

Figure 2 illustrates how collateral requirements affect members’ decision to hedge. For \( F < \hat{F}(I) \) (the pink region), even if all other members hedge, member \( i \) strategically decides not to hedge and engages in risk-shifting. For \( F \geq \hat{F}(I) \) (the blue region), “all hedging” are indeed optimal policies of members.\(^{27}\) Importantly, we show that some members choosing to hedge and others choosing not to hedge is never a possibility. This is because members’ risk-management choices are strategic complements, in line with e.g., Cooper and John (1988) and Farhi and Tirole (2012).

Proposition 2 conveys several key insights. First, members’ incentives for risk-shifting dampen as collateral increases. As such, the CCP can use collateral as a device to align members’ incentives towards the first-best benchmark. To be “incentive-compatible,” collateral needs to be sufficiently high. The right-hand side of Eq. (15) gives the effective

\(^{27}\)For \( F \in [\hat{F}(I), \bar{F}(I)] \), both “all hedging” and “all not hedging” are optimal policies of members. If \( F > \bar{F}(I) \), member \( i \) decides to hedge regardless of others’ decision.
Figure 2. Members’ Risk-management Decisions. This figure illustrates how members’ risk-management choices change with collateral requirements. The vertical and horizontal axes represent initial margin $I$ and default fund $F$. In the pink region where $0 \leq F < \hat{F}(I)$, we have $a(I; F) = r, \forall i$. In the blue region where $\hat{F}(I) \leq F \leq \bar{F}(I)$, we have $a(I; F) = r, \forall i$, or $a(I; F) = s, \forall i$. In the blue region where $\bar{F}(I) < F \leq D - I$, we have $a(I; F) = s, \forall i$. The bolded curve labeled with $IC$ traces the minimum incentive-compatible combinations of collateral types along which efficient risk management among all members is induced.

refundable collateral that is necessary to induce hedging at all members. For a given initial margin $I$, the default fund needs to be higher than the cutoff value $\hat{F}(I)$. Intuitively, a higher default fund contribution means a higher refund to a surviving member. This in turn makes survival more attractive and encourages members to hedge.

Second, both types of collateral are instruments to avert risk-shifting. This is illustrated in Figure 2. The bolded curve, given by, $I + \psi(N - 1; I; \hat{F}) = D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)}p_c$, represents the minimum incentive-compatible combinations of collateral types along which efficient risk management among all members is induced. The required default fund $\hat{F}(I)$ is linearly decreasing in $I$. Importantly, Proposition 2 states that $\partial \hat{F}(I)/\partial I < -1$: When initial margin decreases by one unit, the level of incentive-compatible default fund increases by more than one unit. This is because the externality associated with loss-mutualization makes the default fund more costly to provide hedging incentives to members. Hence, when substituting initial margin with default fund, the CCP needs to collect a higher amount of collateral in total to prevent risk-shifting. This result has important implications: (1) Initial margin and default fund are substitutes, albeit imperfect ones, to induce efficient risk management; and (2) a
unit of initial margin is more cost-effective than a unit of default fund in aligning members’ incentives.

### 3.4 Optimal Collateral Requirements

In this section, we proceed to characterize the optimal collateral requirements in the second-best. Given that the collateral choice affects members’ risk-shifting incentives, the CCP can set collateral so to induce efficient risk management. To pin down the optimal mix of the two collateral types, the CCP balances between the opportunity cost of collateral, its effectiveness in providing incentives, and its cost of recapitalization.

**Definition 1** The optimal collateral requirements set by the CCP, \((I^{SB}, F^{SB})\), maximize the sum of all clearing members’ expected profits, net of the CCP’s expected contribution to the default waterfall. Formally, the CCP solves

\[
\max_{(I,F)} W(I;F) \equiv \max_{(I,F)} \frac{1}{N} \left\{ \sum_i V_i(a(I;F);I;F) - (1 + \beta)p_c \mathbb{E}^a \left[ (N_d(D - I) - N_F)^+ \right] \right\}
\]

subject to

(i) the clearing member’s participation constraint: the expected profit of a member must not be lower than in a bilaterally traded market, i.e.,

\[
V(a(I;F);I;F) \geq V_{BT},
\]

where members’ risk-management decisions \(a(I;F)\) are given by \((14)\), dealers’ expected profits are given by \((11)\) in a centrally cleared market, and by \((10)\) in a bilateral trading market.

(ii) the incentive-compatibility constraint: all members engaging in hedging is consistent with the risk-management decision rule in \((14)\), i.e.,

\[
V(a = s;I;F) \geq V(a = r;I;F).
\]

Different from the first-best benchmark, the CCP designs collateral requirement tak-
ing members’ risk-management decision rule \( a(I; F) \) as given. We refer to the collateral requirements set by the regulator \((I^{SB}, F^{SB})\) as the incentive-constrained optimal collateral. To solve for \((I^{SB}, F^{SB})\), first note that Assumption 1 guarantees that the participation constraint is slack for any of the minimum incentive-compatible combination of collateral, \((I, \hat{F}(I))\). This result is established by Lemma 1.

**Lemma 1** For all \( I \in \left[0, D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}\right] \), it always holds that \( V(a = s; I; \hat{F}(I)) > V_{BT} \). Moreover, \( V(a = s; I; \hat{F}(I)) \) is greater than the value of a dealer with a fully collateralized margin position.

Lemma 1 is intuitive. When the promises to protection buyers are fully guaranteed by the CCP, the dealer earns an additional premium, \( A_{CCP} - A_{BT} \). This premium increases the asset size of the dealer, generating a higher expected profit than in the bilaterally traded market. If the protection buyers are sufficiently risk averse (per Assumption 1), they will pay a sufficiently high premium to dealers so that the latter are better off joining the CCP, despite the collateral requirements. Lemma 1 states that the optimization problem (16) can be recast into finding the most cost-effective combination of \( I \) and \( F \) along the IC curve outlined in Figure 2; that is, the CCP chooses \( I^{SB} = \arg \max_I W(I, \hat{F}(I)) \). The formal result is stated in Proposition 3.

**Proposition 3** The incentive-constrained optimal collateral characterized in Definition 1 is given by \( I^{SB} = D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \) and \( F^{SB} = 0 \).

Proposition 3 states that the incentive-constrained optimal collateral consists only of initial margin. To understand this result, note that the CCP faces a tradeoff between minimizing members’ opportunity cost of collateral and CCP’s expected recapitalization cost. Let us analyze how an additional unit of initial margin impacts this tradeoff. Taking the first-order derivative of the CCP’s objective function \( W(I; \hat{F}(I)) \) with respect to \( I \), we obtain

\[
\frac{\partial}{\partial I} W(I; \hat{F}(I)) = -\beta \frac{\partial}{\partial I} \left(I + \hat{F}(I)\right) - p_c \beta \frac{\partial}{\partial I} \mathbb{E}^s \left[\left(\frac{N_d(D - I)}{N} - \hat{F}(I)\right)^+\right].
\]  

(19)
On the one hand, the CCP aims to reduce a member’s total collateral posting. Proposition 2 establishes that initial margins are more cost-effective in aligning members’ incentives. Formally, \( \frac{\partial}{\partial I} \left( I + \hat{F}(I) \right) < 0 \), that is, an additional unit of initial margin reduces the total collateral a member needs to post. On the other hand, the CCP aims to reduce the recapitalization cost it faces when retrieving end-of-waterfall resources. According to the default waterfall, the CCP has access to the default funds posted by all members if needed, and the initial margin of only the defaulting members. Because of the loss-mutualization mechanism, one unit of default fund is more effective in reducing the CCP recapitalization cost, compared to initial margin. Formally, \( \frac{\partial}{\partial I} \mathbb{E} \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] > 0 \), that is, an additional unit of initial margin increases the CCP’s expected cost of acquiring end-of-waterfall resources.

Whether to use initial margin or default fund depends on which of the two effects described above prevails. We prove in the Appendix that \( \frac{\partial}{\partial I} \left( I + \hat{F}(I) \right) = -\frac{\partial}{\partial I} \mathbb{E} \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] \). Plugging this expression into Eq. (19), we obtain

\[
\frac{\partial}{\partial I} W(I; \hat{F}(I)) = -(1-p_c)\beta \frac{\partial}{\partial I} \left( I + \hat{F}(I) \right) > 0,
\]

which suggests that the optimal collateral rule is to use the maximum possible initial margin, as illustrated in Figure 3; see point A therein. The intuition is the following. Members’ marginal collateral opportunity cost equals the CCP’s recapitalization cost. While both of these marginal costs are equal to \( \beta \), there is a difference between the two: Members’ collateral is prefunded, and thus is posted regardless of whether the credit event occurs. In contrast, the CCP’s recapitalization is only financed ex-post, conditional on the occurrence of the credit event. Given this cost structure, it is cheaper to use end-of-waterfall resources to absorb losses, which makes minimizing members’ ex-ante collateral posting the primary goal in designing collateral. This is why initial margin is preferred.

**Extreme Market Events and the Use of Default Funds.** Does Proposition 3 suggest that default fund is a redundant layer of collateral in the default waterfall? It is quite the opposite. The key reason behind the exclusive use of initial margin is that members’
Figure 3. Optimal Collateral Requirements. This figure illustrates the possible cases for the optimal collateral requirements \((I_{SB}, \hat{F}(I_{SB}))\). The optimal combination is at point \(A\) when \(\beta > p_c \alpha\), at point \(B\) when \(p_c \alpha > \beta\), and at some point along the line between \(A\) and \(B\) when \(\beta = p_c \alpha\).

The marginal opportunity cost of collateral equals the CCP’s recapitalization cost. This makes the recapitalization cost, which is only incurred when the credit event occurs, effectively lower and undermines the value of loss mutualization through default fund. In practice, however, the cost of end-of-waterfall resources is potentially higher than the collateral opportunity cost. To see why, recall that the end-of-waterfall resources are invoked only when multiple members default, depleting all collateral. This happens precisely during periods of market distress when end-of-waterfall resources are scarce to be obtained. Hence, it is likely that the CCP’s ex-post recapitalization cost is higher than the members’ ex-ante opportunity cost of collateral posting. To capture this situation, we model the recapitalization cost as stated in the following assumption.

Assumption 3 The cost of recapitalizing the CCP is linear in the unfunded shortfall and given by \((1 + \alpha)(N_d(D - I) - NF)^+\), where \(\alpha\) is a positive constant.

The next proposition shows that when the CCP faces a sufficiently high recapitalization cost—as is likely the case during periods of systemic distress—allocating more collateral resources to facilitate loss sharing is more valuable.
Proposition 4 Under Assumption 3, the optimal initial margin $I^{SB}$ is given by

$$
\begin{cases}
I^{SB} = D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r-q_s)p_c} & \beta > p_c\alpha \\
I^{SB} \in \left[0, D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r-q_s)p_c}\right] & \beta = p_c\alpha \\
I^{SB} = 0 & \beta < p_c\alpha
\end{cases}
$$

(20)

The optimal default fund $F^{SB}$ is given by

$$
\hat{F}(I^{SB}) = \left(\frac{1 - v^{(N)}_l(q_s)}{u^{(N)}_l(q_s)} + 1\right)(D - I^{SB}) - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r-q_s)p_cu^{(N)}_l(q_s)},
$$

(21)

where $v^{(N)}_l(q_s) \in (0, 1)$, $u^{(N)}_l(q_s) > 0$, and the explicit expressions of these two terms are given by (A5) and (A6) in the Appendix. Furthermore, we have that $\hat{F}\left(I^{SB} = D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r-q_s)p_c}\right) = 0$.

Proposition 4 shows how the optimal allocation of collateral depends on the relation between $\alpha$ and $\beta$. Proposition 3, in which $\alpha = \beta$, falls from the expression of $I^{SB}$ given in (20) when $\beta > p_c\alpha$. If the expected cost of recapitalizing the CCP is higher than posting collateral at the margin ($p_c\alpha > \beta$), reducing recapitalization costs is the dominating factor, and a stronger weight is placed on strengthening loss mutualization through default funds. Hence, decreasing the initial margin by an additional unit always improves the total value. This corresponds to setting $I^{SB} = 0$ at point $B$ in Figure 3. Finally, if collateral and expected end-of-waterfall resources are equally costly to finance ex-ante ($\beta = p_c\alpha$), whether the members or the CCP bear more cost does not affect social welfare. Hence, the combination of collateral is optimal as long as it is incentive compatible. This corresponds to any convex combination of points $A$ and $B$ in Figure 3.

Unlike most studies on central clearing that focus either on initial margin or default funds, our model solves for the optimal initial margin and default fund jointly. The optimal mix of collateral resources features the fundamental trade-off between minimizing CCP’s recapitalization cost through ex-post loss-sharing, and minimizing members’ collateral cost while preventing ex-ante risk-shifting. Our model, therefore, provides insights on the relative
costs and benefits of each layer of the CCP default waterfall structure.

**Optimal Cover Rule for Default Funds.** When the cost of recapitalizing a CCP during stressed events is high, Proposition 4 offers a rationale for collecting default funds. Next, we elaborate on the optimal default fund contribution given in Eq. (21). Our objective is to compare the default fund level predicted by our model with the current CPSS-IOSCO (2012) international regulatory guideline known as the “Cover 2” rule. According to the “Cover 2” rule, the total default funds posted by members should cover at least the shortfalls of the two largest clearing members, i.e., \( NF \geq 2(D - I) \).

We define a generalized “Cover x\%” rule for a given number \( N \) of members and initial margin \( I \),

\[
x(I; N) = \frac{\hat{F}(I; N)}{D - I},
\]

where \( \hat{F}(I; N) \) is the optimal default fund required to incentivize members to hedge given in Eq. (21). The implied optimal cover number is \( Nx(I; N) \). When \( Nx(I; N) > 2 \), our model provides a rationale to charge a default fund larger than the current regulatory requirement. Interestingly, we find that while the optimal cover number depends on \( N \), the optimal “Cover x\%” rule shows little variation with \( N \), especially when \( N \) is larger than 15; see Figure 4 for an illustration.\(^{28}\) For this reason, the “Cover x\%” rule is robust against entry and exit of members in the clearing business, whereas the “Cover 2” rule is not.\(^{29}\) The next proposition characterizes the asymptotic behavior of the “Cover x\%” rule as the number of members grows large.

**Proposition 5** In the limiting case of a large clearing network—as \( N \to \infty \)—the optimal

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\(^{28}\) The sum of independent binomial distributions converges to normal distribution fairly quickly as \( N \) increases, an implication of the central limit theorem.

\(^{29}\) Since the enactment of the Dodd-Frank Act of 2010, several dealers entered the clearing business. Exit events, often due to high operational costs, are also frequent. For example, in May 2014 the Royal Bank of Scotland announced the wind-down of its clearing business. State Street, BNY Mellon, and Nomura followed suit, with each of them shutting down part or all of their clearing business.
Figure 4. Optimal “Cover x%” Rule. This figure shows how the optimal cover number $Nx(I; N)$ (the left panel) and the “Cover x%” rule $x(I; N)$ (the right panel) vary with the number $N$ of clearing members. In this example, the default funds should cover payment shortfalls caused by 25.9% of members’ defaults. Model parameters: $q_r = 0.1, q_s = 0.05, R_r = 3, R_s = 2.85, \gamma = 2, \beta = 0.1, \alpha = 0.4, p_c = 0.3, D = 3,$ and $N = 30$. The optimal default fund is $F^{SB} = F(I^{SB} = 0) = 0.776$. We set $N = 30$, consistently with number of clearing members of ICE Clear Credit, the largest CDS clearinghouse.

“Cover x%” rule admits the following explicit expression:

$$x(I^{SB}; N) \to 1 - \frac{(1 - q_s)(\mu_s - \mu_r)D(1 + \gamma D(1 - p_c))}{(q_r - q_s)(D - I^{SB})}. \quad (23)$$

The optimal “Cover x%” rule is a positive fraction greater than $q_s$; it decreases with buyers’ risk aversion $\gamma$ and the expected return differential $\mu_s - \mu_r$, and increases with the probability of the credit event $p_c$.

As the number of clearing members increases, the optimal cover rule takes a simple form—rather than covering a fixed number of clearing members as prescribed by “Cover 2,” the total default funds should cover a fixed fraction of members.\(^{30}\) The closed-form expression in Proposition 5 makes it possible to compute the sensitivity of the optimal cover ratio to various model parameters. First, the optimal “Cover x%” rule decreases with buyers’ risk aversion.

As protection buyers become increasingly risk averse, they value more the benefits of central

\(^{30}\)Our analysis is not aimed at criticizing the Cover 2 per se. In fact as Murphy and Nahai-Williamson (2014) argue, Cover 2 has its practical merit especially when we consider heterogeneity of a CCP’s exposure to its members, a feature that we will explore further in Section 4.2.
clearing and are charged with a higher CDS price. This scales up dealers’ investment, makes survival more attractive, and reduces dealers’ incentives for risk-shifting. Second, “Cover x%” decreases with the expected return differential. As risk management becomes more appealing, members’ incentives for risk-shifting fall, so a lower default fund is required to align the incentives. Third, an increase in the probability of the credit event raises the expected liability of a surviving member. A reduced profit upon survival exacerbates risk-shifting incentives and requires more default funds to counteract this effect.31

4 Robustness Analysis

In this section, we examine the robustness of our model predictions with respect to two alternative settings that incorporate realistic features of the clearing business.

4.1 Convex CCP Recapitalization Cost

In the baseline model, we assume that the recapitalization cost of the CCP, incurred after collateral resources are exhausted, is linear in the shortfall. As both the CCP’s recapitalization cost and members’ opportunity cost of collateral are linear, our baseline model shows that the trade-off between initial margin and default fund boils down to comparing the relative magnitude of these two marginal costs. Such a setup allows us to illustrate the economic trade-off in a transparent manner. Nevertheless, when multiple defaults occur and all prefunded collateral resources are exhausted, the market is likely distressed, where access to end-of-waterfall resources becomes increasingly costly. To capture this market characteristic, we adapt Assumption 3 from the baseline model and assume a convex structure for the CCP recapitalization cost.

31The comparative static with respect to $p_c$ is related to that of Biais et al. (2016). They show that bad news about the risk of an asset underlying a derivative (corresponding to an increase in $p_c$ in our model) increases protection sellers’ expected liability and undermines their risk-prevention incentives. However, the objective of our study is different from Biais et al. (2016). They focus on the effect of an increase in $p_c$, and show that the optimal contract to eliminate moral hazard may require margin calls after the bad news; they do not model default funds. In our model, $p_c$ is fixed, and we study the ex-ante optimal mix of initial margins and default fund requirements.
Figure 5. Robustness Analysis: Convex Recapitalization Cost. This figure illustrates the optimal collateral requirements when the CCP’s recapitalization cost is convex. The optimal collateral features interior levels of initial margin and default fund at point $A$.

Assumption 3-C The cost of recapitalizing the CCP is convex in the unfunded shortfall and given by $(N_d(D - I) - NF)^+ + \alpha ((N_d(D - I) - NF)^+)^2$.

We show that the trade-off between initial margin and default fund highlighted in the baseline model continues to hold. Moreover, the nonlinearity of CCP’s recapitalization cost allows us to pin down interior levels of initial margin and default fund. The following proposition is the counterpart of Proposition 4.

**Proposition 6** Under Assumption 3-C, the optimal initial margin $I^{SB}$ is given by

$$I^{SB} = \begin{cases} \arg \max_{I} W(I; \hat{F}(I)), & I \neq \emptyset \\ \arg \max_{0, D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_e}} W(I; \hat{F}(I)), & I = \emptyset \end{cases}$$

where $\mathcal{I} = \{I^*_l : I^*_l \in (I_{l+1}, I_l), l = 0, ..., N - 1\}$, the kinks are given by $\{I_l : I_l > I_{l+1}, l = \frac{N F(I_l)}{D - I_l}\}$, and $I^*_l$ is the unique solution to $\frac{\partial}{\partial I} W(I; \hat{F}(I)) = 0$ for each $l$; The optimal default fund $F^{SB}$ is given by Eq. (21).

Proposition 6 characterizes the optimal collateral requirements under a convex CCP recapitalization cost. Compared with Proposition 4 in which the optimal initial margin takes a corner solution, here the initial margin takes an interior value, suggesting that both collateral
instruments—initial margin and default fund—are necessary; see Figure 5. To illustrate the economic intuition, let us consider the case of small $\alpha$. If $\alpha$ is small, reducing members’ opportunity cost of collateral is the dominating force in the CCP’s objective function; hence, increasing the fraction of collateral allocated as initial margin improves the total value. However, as more collateral is allocated into initial margin and subtracted from default fund, the CCP has fewer resources to mutualize losses. This increases the expected need of end-of-waterfall resources and, under the convex cost structure, endogenously raises the marginal cost of CCP recapitalization. Such an endogenous response creates a counterforce that calls for an increase in default fund, until the marginal value of increasing initial margin goes to zero.

4.2 Heterogeneity in Size

In our baseline model, we assume that the CCP has the same exposure, $D$, to each clearing member. In practice, CCPs’ outstanding exposures are likely to be concentrated in one or a few large clearing members. According to Office of Financial Research (2017), the largest five clearing members account for 44 to 66 percent of average daily open positions in CME, ICE Clear Credit, and Options Clearing Corporation, and the share is 28 percent for LCH. The fractions of margin and default fund contributions by the largest five clearing members are also substantial.

Motivated by these empirical observations, we extend our model to account for heterogeneity in exposures among members. We adapt the baseline setup as follows. One clearing member (say member $i = 1$) is bigger than the rest, while all other members are ex-ante identical. Specifically, the big member is $K$ times ($K > 1$) the size of other clearing members in terms of the size of CDS contracts sold and asset investment. Hence, the size-$K$ member has expected payoff from risky asset investment equal to $KA_{\text{CCP}}\mu_a$ and an obligation to CDS buyers equal to $KD$. The protection buyers are not affected by this change. To investigate whether members should post collateral disproportionately to their size, let us denote the initial margin and default fund of the big member as $KI_{\text{big}}$ and $KF_{\text{big}}$, respectively.
In this setup, we show that the economic forces highlighted in the baseline model are qualitatively robust (albeit the analysis becomes more tedious). Like in the baseline model, posting collateral increases members’ pledgeable income and enables them to credibly commit to hedging. Default fund allows for loss mutualization through the CCP’s default waterfall mechanism; hence, a unit of default fund may be used to cover other members’ default, thereby contributing less to the pledgeable income of the posting member. For this reason, between the two collateral instruments, initial margins are more cost-effective in aligning members’ incentives for risk management, for both the big and the small members. The next proposition is the counterpart of Proposition 2.

Proposition 7 Suppose that the size of member \( i = 1 \) is \( K \) times \((K > 1)\) that of other members. Given collateral requirements \( I \) and \( F \), the members’ risk-management decisions satisfy

\[
\begin{align*}
  a_i(I; F) &= r, \forall i, \quad F < \hat{F}_{big}(I) \\
  a_1(I; F) &= s, a_i(I; F) = r, \forall i \geq 2, \quad \hat{F}_{big}(I) \leq F < \hat{F}_{small}(I) \\
  a_i(I; F) &= s, \forall i, \quad \hat{F}_{small}(I) \leq F < D - I
\end{align*}
\]

where the expressions of \( \hat{F}_{big}(I) \) and \( \hat{F}_{small}(I) \) are given in the Appendix. Moreover, we have

\[
0 \leq \hat{F}_{big}(I) \leq \hat{F}_{small}(I) < D - I; \text{ both } \hat{F}_{big}(I) \text{ and } \hat{F}_{small}(I) \text{ are piecewise linear and strictly decreasing functions of } I, \text{ with } \frac{\partial \hat{F}_{small}(I)}{\partial I} < \frac{\partial \hat{F}_{big}(I)}{\partial I} < -1.
\]

As in the baseline model, members have incentives to take excessive risk, and the incentives for risk-shifting dampen with the level of collateral posted. Additionally, one unit of initial margin is more effective in aligning members’ incentives, as seen from the fact that both \( \hat{F}_{big}(I) \) and \( \hat{F}_{small}(I) \) have slopes less than \(-1\).

Different from the baseline model, however, the minimum incentive-compatible combinations of collateral differ by member size. Should one be more concerned about the risk-shifting of big members or small members? From Proposition 7, we have \( \hat{F}_{big}(I) \leq \hat{F}_{small}(I) \), where the total default fund posted by the big member is \( K\hat{F}_{big}(I) \). This suggests that for a given initial margin, the default fund required to induce efficient risk management is disproportion-
Figure 6. Robustness Analysis: Heterogeneity in Size. This figure illustrates the optimal collateral requirements when the CCP has heterogeneous exposures to members. The minimum incentive-compatible combinations of collateral vary depending on the member’s size: They are labeled by $IC_{big}$ for big members and $IC_{small}$ for small members.

Proposition 8 Suppose that the size of member $i = 1$ is $K$ times ($K > 1$) that of other members. Under Assumption 3, the optimal initial margin $I^{SB}$ is given by

$$
\begin{cases}
I_{big}^{SB} = I_{small}^{SB} = D - \frac{ACC_P(\mu_s - \mu_r)}{(q_r - q_s)p_c} & \beta > p_c \alpha \\
I_{big}^{SB} = I_{small}^{SB} \in [0, D - \frac{ACC_P(\mu_s - \mu_r)}{(q_r - q_s)p_c}] & \beta = p_c \alpha \\
I_{big}^{SB} = I_{small}^{SB} = 0 & \beta < p_c \alpha
\end{cases}
$$

(26)

The optimal default fund is $K\hat{F}_{big}(I_{big}^{SB})$ for a big member and $\hat{F}_{small}(I_{small}^{SB})$ for a small member, where $\hat{F}_{big}\left(I_{big}^{SB} = D - \frac{ACC_P(\mu_s - \mu_r)}{(q_r - q_s)p_c}\right) = \hat{F}_{small}\left(I_{small}^{SB} = D - \frac{ACC_P(\mu_s - \mu_r)}{(q_r - q_s)p_c}\right) = 0$.

When the cost of recapitalizing the CCP is significantly high ($p_c \alpha \geq \beta$), default fund is adopted as a collateral instrument. The optimal default fund collection should follow the generalized “Cover x%” rule as discussed in Section 3.4. As the number of members $N$ grows large, the optimal “Cover x%” rule shows little variation with $N$. Hence, in the limiting case, the optimal default funds collected should cover the shortfalls of a fraction, rather than of a fixed number, of clearing members. This conclusion also holds if there is more than one big
member, as long as the total mass of big members remains zero. Below is the counterpart of Proposition 5.

**Proposition 9** In the limiting case of a large clearing network—as \( N \to \infty \) and \( K/N \to 0 \)—the optimal “Cover x%” rule for collecting default funds, \( x(I^{SB}; N) \), admits the same expression given by Eq. (23).

Taken together, the results above show that our characterization of the optimal collateral requirements carries through to a setting with heterogeneous member size. The prediction that the required collateral is *disproportionately lower* for the big member stands in contrast with that from the banking literature, according to which big institutions particularly tend to take excessive risk. For example, Davila and Walther (2019) show that large banks have higher leverage and default more frequently, because they internalize the fact that their decisions directly affect bailout policies. In their setting, taxpayers bear the negative externalities. In our setting of central clearing, however, members impose negative externalities on each other. The big member contributes a larger amount to the total default fund pool and effectively acts as an *internally coordinated group* of \( K \) small members. This big member thus finds it easier to internalize the externality and undertakes efficient risk management.\(^{32}\) In contrast, the small members free ride on the big member in sharing losses, and have a stronger incentive to shift risk.

5 Policy and Empirical Implications

We discuss policy implications of our results, and list empirical predictions implied by our analysis.

5.1 Policy Implications

**Regulation of collateral requirements.** As the volume of centrally cleared derivatives grows, so does the importance of regulating them. The new international guidelines adopted

\(^{32}\)Consider, for example, the extreme case of a single big member and zero small members. The single big member fully internalizes any externality, making initial margin and default fund perfect substitutes.
since 2016 require CCPs to make quarterly disclosures of their default waterfalls (CPMI-IOSCO, 2015), and hence make it feasible to closely monitor CCP collateral requirements (Office of Financial Research, 2017). While comparable waterfall structures are adopted across the board, the quarterly filings of CCPs in accordance with these recent international guidelines reveal significant variation in how resources are allocated in initial margin and default funds.\textsuperscript{33}

Our framework allows to assess the trade-off between different collateral layers of the default waterfall. We show that, members’ collateral opportunity cost and CCP recapitalization cost are important factors in designing collateral requirements. When the opportunity cost of collateral is the primary concern, the collateral policy should rely more on initial margin. This happens, for example, in a high-interest-rate environment during economic expansions. In contrast, when concerns about the CCP’s recapitalization dominate, the collateral policy should rely more on default fund. This applies to an environment with high cost of CCP recapitalization, for instance a market with an inverted yield curve. Our analysis establishes a rationale for the collection of default funds in such a market scenario, and prescribes a generalized “Cover x\%” rule: The total default funds collected should cover the shortfalls of a fraction of clearing members. In addition, our findings highlight the importance of accounting for members’ size when allocating collateral requirements. Interestingly, our model predicts that big members should contribute disproportionately lower levels of default fund relative to small members.

**CCP resilience.** To cover losses in excess of prefunded collateral, the CCP has recourse to end-of-waterfall resources, which presumably is invoked during extreme market events when multiple members may default simultaneously. In these stressed markets, the recapitalization cost is likely to be high due to low market liquidity, which puts CCP’s resilience at stake. Hence, one can alternatively view the CCP having to use end-of-waterfall resources as a

\textsuperscript{33}Using proprietary data from CCPView, Paddrik and Zhang (2019) document that the average ratio between default fund to initial margin varies from 17\% to 30\% across different derivative classes and from 0 to 34\% across regions. For instance, the ratio of default fund to initial margin for CDS is 40.9\% at CME and 7.1\% at ICE Clear Credit.
severe challenge to its resilience. In the context of our model, a high recapitalization cost ($\alpha$) in the CCP’s objective function can thus be interpreted as a strong regulatory weight on reducing systemic distress. Our results indicate that default fund is a useful tool to ensure CCP resilience by mutualizing losses. Our results also inform the aggregate amount of default fund collection. Under the current “Cover 2” rule, the CCP needs to resort to end-of-waterfall resources as soon as more than two clearing members default, hence challenging its resilience. Charging a default fund beyond “Cover 2” would lower the probability of CCP recapitalization. In fact, in a CCP consisting of many members and under the optimal collateral requirements, the expected recourse to end-of-waterfall resources would be zero. This policy implication is a direct consequence of the following proposition.

**Proposition 10** In the limiting case of a large clearing network, the expected losses at the CCP under the optimal collateral requirements ($I^{SB}, \hat{F}(I^{SB})$) converge to 0, i.e.,

$$\lim_{N \to \infty} \mathbb{E}^s \left[ \left( \frac{N_d}{N}(D - I^{SB}) - \hat{F}(I^{SB}) \right)^+ \right] = 0.$$

**The value of central clearing.** Our study highlights the value of central clearing when proper regulation is in place. In the constrained-optimal allocation, dealers are better-off clearing with the CCP because they earn higher profits than in bilateral trades. The underlying reason is that, by pooling resources against default, the CCP fully guarantees the promised payments to protection buyers. Consequently, dealers earn an additional insurance premium. However, if the collateral requirements set by the CCP deviate from the constrained-optimal allocation, or if the dealers are not able to receive an additional premium for eliminating the counterparty risk, then dealers might prefer not to join the CCP. These findings offer regulatory guidance on how to encourage clearing participation. First, the clearing mandate should be accompanied by dealers’ ability to charge a higher spread for a centrally cleared contract. For this to happen, the counterparty risk needs to be materially reduced in a centrally-traded platform, which requires the CCP to collect enough resources to guarantee the transactions. Second, the collateral requirements should be thoughtfully
regulated. While providing incentives to encourage risk management is essential, it must be
done in a way that accounts for members’ opportunity cost of collateral.

5.2 Empirical Implications

According to our theory, posting collateral increases members’ pledgeable income and
provides them incentives for risk management. To be “incentive-compatible,” the collateral
needs to be sufficiently high. Based on Proposition 2, the minimum total refundable collateral
is given by $D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}$ where $A_{CCP}$ is the price of a centrally cleared CDS given by
Eq. (7).\footnote{Plugging in the expression for $A_{CCP}$, we obtain that the total required collateral has an expression of
$D \left( 1 - (1 + \gamma D(1 - p_c)) \frac{\mu_s - \mu_r}{q_r - q_s} \right)$. This term is concave with respect to $D$, is increasing with $p_c$, and decreasing
with $\gamma$, $A_{CCP}$, and $\frac{\mu_s - \mu_r}{q_r - q_s}$.} We conclude that the total needed collateral is increasing with the risk of the
credit event ($p_c$) and decreasing with the CDS price ($A_{CCP}$) as well as the effectiveness of
risk-management procedures ($\frac{\mu_s - \mu_r}{q_r - q_s}$).

**Empirical Implication 1** Other things equal, the CCP collects a higher amount of collateral when the reference entities underlying the CDS contracts impose a higher credit risk, and collects a lower amount of collateral when the spread of a centrally cleared CDS contract is high. Clearing members who demonstrate more effective risk-management procedures post lower amounts of collateral.

The central insight of the model is that the optimal mix between initial margin and default
fund depends on the magnitude of the CCP’s recapitalization cost relative to members’
opportunity cost of collateral. Hence, the allocation of collateral resources in the default
waterfall depends on the current macroeconomic conditions. For example, default funds
should be higher in an environment characterized by flight-to-safety and high probability of
default clustering. The following predictions are supported by Propositions 4 and 6.

**Empirical Implication 2** Other things equal, the fraction of collateral allocated in initial
margins increases with the collateral opportunity cost and decreases with measures of systemic
distress.
The following empirical predictions follow from Propositions 4 and 8.

**Empirical Implication 3** The total pool of default funds at a CCP increases with the entry of clearing members. Clearing members who maintain larger open positions at a CCP post more collateral, but their default fund contributions are disproportionately lower than other members.

All the empirical implications above share a common caveat: the CCP acts as a benevolent social planner who sets the optimal collateral requirements to maximize the total value of market participants. However, this is not always the case in practice. In fact, as Huang (2019) points out, some CCPs are for-profit publicly listed financial firms and thus may have misaligned incentives. They may, for instance, act strategically in setting collateral to maximize their fee income, hence setting requirements which deviate from the optimal benchmark.

6 Conclusion

Reforms after the financial crisis of 2007-09 promote the use of central counterparties (CCPs) to reduce counterparty risk. To ensure its resilience, a CCP collects two types of collateral from clearing members: initial margin and default funds. Despite extensive debate on current clearing practices, there is limited work aimed at understanding the design and regulation of collateral at CCPs, and especially on how to determine margins and default fund jointly.

Our paper fills this gap and provides a tractable framework to study the optimal levels of initial margin and default fund requirements. Posting collateral increases members’ pledgeable income, thereby reducing their risk-shifting incentives. However, different types of collateral have distinct implications for members’ risk-shifting incentives and CCP’s resilience. Initial margin is more cost-effective in aligning members’ incentives. By contrast, default fund is less effective in providing incentives ex-ante because the collateral resources are pooled across members to achieve loss-sharing ex-post. This loss-mutualization benefit
is valuable for the CCP’s resilience. Our findings generate novel implications for regulating collateral requirements at clearinghouses and enhancing their resilience. While our model is focused on central clearing, the framework and economic trade-off can be adapted more broadly to other markets in which risk-shifting poses a concern in the face of pooled resources.

The tractability of our model opens the door to several directions for future research. First, besides heterogeneity in size, it would be of interest to explore how the results generalize to a setting in which members differ ex-ante in balance sheet conditions, such as asset liquidity. Second, we have assumed that the investment outcomes are independent across dealers. One may introduce correlated investments caused, for example, by securitization. As Biais et al. (2012) show, when aggregate risk is significant, protection buyers should not be fully insured against counterparty risk. We conjecture that the presence of aggregate risk might change the importance of loss-sharing relative to collateral opportunity cost, and introduce an additional layer of trade-off between initial margin and default funds.
References


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Appendix: Proofs

Proof of Proposition 1

The CCP’s optimization problem in the first-best benchmark is

\[
\max_{(a; I; F)} W^{FB}(a; I; F) \equiv \max_{(a; I; F)} \frac{1}{N} \left\{ \sum_{i} V(a_i, a_{-i}; I; F) - (1 + \beta)p_c \mathbb{E}^{g} \left[ \left( N_d(D - I) - NF \right)^+ \right] \right\}
\]

\[
= \max_{(a; I; F)} A_{CCP} \mu_a - \beta (I + F) - p_c \mathbb{E}^{g} \left[ \left( N_d(D - I) - F \right)^+ \right] - p_c D.
\]

It follows from conditions (1)–(2) that \( W^{FB}(s; I; F) > W^{FB}(r; I; F) \), so \( a_i^{FB} = s, \forall i \). We further take first-order conditions with respect to \( I \) and \( F \):

\[
\frac{\partial W^{FB}(s; I; F)}{\partial I} = -\beta + p_c \beta q_s \mathbb{E}^{s} \left[ \frac{N_d(D - I)}{N} - F > 0 \right] < 0,
\]

\[
\frac{\partial W^{FB}(s; I; F)}{\partial F} = -\beta + p_c \beta q_s \mathbb{E}^{s} \left[ \frac{N_d(D - I)}{N} - F > 0 \right] < 0,
\]

where \( \mathbb{E}^{s}[\cdot] \) denotes the probability conditional on the credit event and the risk management choice \( a = s \). This proves that collateral is zero in the first-best benchmark, i.e., \( I^{FB} = F^{FB} = 0 \).

Proof of Proposition 2

To understand the risk-management decision of member \( i \), let us consider different scenarios of other members’ decisions. Given collateral requirements \( I \) and \( F \), and assuming \( g = 0, \ldots, N - 1 \) of the remaining \( (N - 1) \) members choose to hedge, member \( i \) chooses to hedge if and only if \( V(s, |a_{-i} = s| = g; I; F) \geq V(r, |a_{-i} = s| = g; I; F) \), that is,

\[
I + \psi (g; I; F) \geq D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c}, \tag{A1}
\]

where the expected refund of default fund \( (\psi) \) is given by Eq. (13).

Define \( \hat{F}(I) \) and \( \bar{F}(I) \) as the cutoff values that satisfy Eq. (15). Lemma IA.1 in the Internet Appendix shows that the function \( \psi (g; I; F) \) is piecewise linear in both \( I \) and \( F \), and strictly increasing in \( g, I, \) and \( F \). Hence, we have that \( \hat{F}(I) \) and \( \bar{F}(I) \) are uniquely determined for a given \( I \). Since \( \psi(N - 1; I; \bar{F}) = \psi(0; I; \bar{F}) \), and \( \psi(0; I; D - I) = D - I \), we have \( \hat{F}(I) < \bar{F}(I) < D - I \). We can distinguish the following cases:

1. \( I + \psi (N - 1; I; F) < D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c} \), which is equivalent to \( 0 \leq F < \hat{F}(I) \) by Lemma IA.1. In this case, each member chooses not to hedge regardless of other members’ choices.

2. \( I + \psi (0; I; F) > D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c} \), which is equivalent to \( \bar{F}(I) < F < D - I \). In this case, each member chooses to hedge regardless of other members’ choices.
3. \( I + \psi(0; I; F) \leq D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \leq I + \psi(N - 1; I; F) \), which is equivalent to \( \hat{F}(I) \leq F \leq \hat{F}(I) \). In this case, all choosing to hedge and all choosing not to hedge are both optimal choices of members. We next show that no other risk management profile can be optimal. We argue by contradiction. Suppose \( g \) members choose to hedge and \( (N - g) \) members choose not to hedge, for some \( g = 1, \ldots, N - 1 \). Then, any member choosing not to hedge faces \( g \) members choosing to hedge and \( (N - g - 1) \) members choosing not to hedge. Using inequality (A1), this member chooses not to unilaterally deviate to hedge only if

\[
I + \psi(g; I; F) \leq D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}. \tag{A2}
\]

Using an analogous reasoning, any member choosing to hedge faces other \( (g - 1) \) members who choose to hedge and \( (N - g) \) members who choose not to hedge. Using inequality (A1), for this member to not deviate from his hedge choice it must hold that

\[
I + \psi(g - 1; I; F) \geq D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}. \tag{A3}
\]

But conditions (A2) and (A3) cannot hold simultaneously because \( \psi(g - 1; I; F) < \psi(g; I; F) \) (see Lemma IA.1). Hence, we obtain a contradiction.

Combining the above three cases, we obtain Eq. (14) and Eq. (15).

To analyze the term \( \hat{F}(I) \) (and \( \hat{F}(I) \)), we start from Eq. (15) and solve for \( \hat{F}(I) \). Using the definition of \( \psi(N - 1; I; F) \) in Eq. (A25) of Lemma IA.1, we have

\[
\psi(N - 1; I; F) = \sum_{k=N-1-l}^{N-1} \binom{N-1}{k}(1 - q_s)^k q_s^{N-1-k}(F - \frac{N - (k + 1)}{k + 1}(D - I - F)) = \sum_{k=N-1-l}^{N-1} \binom{N-1}{k}(1 - q_s)^k q_s^{N-1-k}(1 - \frac{N}{k + 1})(D - I) + \frac{N}{k + 1}F
\]

\[
= \left( v_l^{(N)}(q_s) - u_l^{(N)}(q_s) \right) (D - I) + u_l^{(N)}(q_s)F, \tag{A4}
\]

where \( l = \lfloor \frac{NF}{D - I} \rfloor \) is the maximum number of defaults so that the collateral resources are not depleted, and \( v_l^{(N)}(q_s) \) and \( u_l^{(N)}(q_s) \) are given by

\[
v_l^{(N)}(q_s) = \sum_{k=N-1-l}^{N-1} \binom{N-1}{k}(1 - q_s)^k q_s^{N-1-k} = \sum_{k=0}^{l} \binom{N-1}{k} q_s^k (1 - q_s)^{N-1-k}; \tag{A5}
\]

\[
u_l^{(N)}(q_s) = \sum_{k=N-l}^{N} \binom{N}{k}(1 - q_s)^{k-1} q_s^{N-k} = \frac{1}{1 - q_s} \sum_{k=0}^{l} \binom{N}{k} q_s^k (1 - q_s)^{N-k}. \tag{A6}
\]

To understand the terms \( v_l^{(N)}(q_s) \) and \( u_l^{(N)}(q_s) \), suppose \( X \) follows a Binomial distribution with parameter \( (N - 1, \alpha) \) and \( Y \) follows a Binomial distribution with parameter \( (N, \alpha) \), then from Eq. (A5)–(A6), we have \( v_l^{(N)}(a) = \Pr(X \leq l) \) and \( u_l^{(N)}(a) = \frac{\Pr(Y \leq l)}{1 - \alpha} \). Hence, both \( v_l^{(N)}(q_s) \) and \( u_l^{(N)}(q_s) \) are step functions of \( l \), \( v_l^{(N)}(q_s) \in (0, 1) \), and \( u_l^{(N)}(q_s) > 0 \).
Plugging Eq. (A4) into Eq. (15) and solving for \( \hat{F}(I) \), we obtain

\[
\hat{F}(I) = \left( \frac{1 - v_i(N)(q_s)}{u_i(N)(q_s)} + 1 \right) (D - I) - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c u_i(N)(q_s)}. \tag{A7}
\]

The properties of \( v_i(N)(q_s) \) and \( u_i(N)(q_s) \) suggest that \( \hat{F}(I) \) is piecewise linear and strictly decreasing in \( I \), with

\[
\frac{\partial}{\partial I} \hat{F}(I) = - \left( \frac{1 - v_i(N)(q_s)}{u_i(N)(q_s)} + 1 \right) < -1.
\]

The properties of \( \tilde{F}(I) \) can be examined analogously by replacing \( q_s \) with \( q_r \) in (A5)–(A7).

Finally, it follows from Eq. (15) that \( \psi(N - 1; I; \hat{F}) = \psi(0; I; \hat{F}) = 0 \) if \( I = D - \frac{A_{CCP}(\mu_r - \mu_s)}{(q_r - q_s)p_c} \). Since \( \psi(g; I; F) > 0 \) for \( F > 0 \) by Lemma IA.1, we must have that \( \hat{F} \left( D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \right) = \tilde{F} \left( D - \frac{A_{CCP}(\mu_r - \mu_s)}{(q_r - q_s)p_c} \right) = 0 \). This completes the proof of Proposition 2.

**Proof of Lemma 1**

Combining Eq. (11), (13), and (15), we obtain

\[
V(a = s; I; \hat{F}(I)) = - (\beta + p_c)(I + \hat{F}(I)) + (1 - q_s) \left( A_{CCP} R_s - p_c(D - I - \psi(N - 1; I; \hat{F}(I)) \right) \\
= - (\beta + p_c)(I + \hat{F}(I)) + A_{CCP} \mu_s - (1 - q_s)p_c \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}. \tag{A8}
\]

Using that \( \frac{\partial}{\partial I} \hat{F}(I) < -1 \) by Proposition 2, we conclude that

\[
\frac{\partial}{\partial I} V(a = s; I; \hat{F}(I)) = - (\beta + p_c) \left( 1 + \frac{\partial}{\partial I} \hat{F}(I) \right) > 0.
\]

The above inequality means a clearing member prefers the use of initial margin in any incentive-compatible combination of collateral types, i.e.,

\[
V(a = s; I; \hat{F}(I)) \geq V(a = s; I = 0; \hat{F}(0)). \tag{A9}
\]

Evaluating Eq. (A8) at \( I = 0 \), we have

\[
V(a = s; I = 0; \hat{F}(0)) = - (\beta + p_c) \hat{F}(0) + A_{CCP} \mu_s - (1 - q_s)p_c (D - \psi(N - 1; 0; \hat{F}(0)).
\]

Since \( \hat{F}(0) < D \), using results from Lemma IA.2 we conclude that

\[
V(a = s; I = 0; \hat{F}(0)) > V(a = s; I = 0; F = D) = - (\beta + p_c)D + A_{CCP} \mu_s. \tag{A10}
\]

When a member has a fully collateralized position (either using initial margin, or default fund, or a combination of both), we have \( I + \hat{F} = D \). In this case, the protection buyer faces zero counterparty risk, implying a CDS price equal to \( A_{CCP} \), and the member chooses \( a = s \).
Hence, the value of a member with a fully collateralized position is

$$V(a = s; I; D - I) = -(\beta + p_c)D + A_{CCP}\mu_s < V(a = s; I = 0; \hat{F}(0)) \leq V(a = s; I; \hat{F}(I)), \quad (A11)$$

where above, the first equality follows from the fact that $$\psi(N - 1; I; D - I) = D - I$$ (proven in Lemma IA.1), and the inequalities follows in a straightforward manner from (A9)–(A10).

Finally, Assumption 1 imposes the restriction $$\gamma > \bar{\gamma}$$, where the explicit expression of $$\bar{\gamma}$$ is given by

$$\bar{\gamma} = \frac{\beta + (1 - \mu_r)p_c q_s - p_c (\mu_s - \mu_r)}{(1 - p_c)(\mu_s - \mu_r) + (1 - q_r p_c)q_r \mu_r) p_c D}. \quad (A12)$$

Plugging Eq. (A12) into $$\gamma > \bar{\gamma}$$, we obtain

$$A_{CCP}\mu_s - A_{BT}\mu_r \geq (\beta + q_r p_c)D \iff V(a = s; I; D - I) \geq V_{BT}, \quad (A13)$$

where $$V_{BT}$$ is given in Eq. (10).

Combining (A11) and (A13), we conclude that the value of a clearing member under the minimum incentive-compatible combination of collateral ($$I, \hat{F}(I)$$) is greater than the value under a fully collateralized position, and is greater than the value in a bilateral trade with zero collateral.

**Proof of Proposition 3**

Plugging $$\hat{F}(I)$$ into the objective function $$W(I; \hat{F}(I))$$, and taking the derivative with respect to $$I$$, we obtain Eq. (19). Moreover, using Lemma IA.3, we obtain

$$\frac{\partial}{\partial I} \left[ I + \hat{F}(I) \right] = -\frac{\partial}{\partial I} \mathbb{E}^s \left[ \left( \frac{N_d(D - I)}{N} - \hat{F}(I) \right)^+ \right] = -\frac{1 - v_l^{(N)}(q_s)}{u_l^{(N)}(q_s)}.$$

Together with Eq. (19), the above equations imply

$$\frac{\partial}{\partial I} W(I; \hat{F}(I)) = \beta(1 - p_c) \frac{1 - v_l^{(N)}(q_s)}{u_l^{(N)}(q_s)} > 0,$$

where the last inequality follows from the fact that $$v_l^{(N)}(q_s) \in (0, 1)$$ and $$u_l^{(N)}(q_s) > 0$$ (shown in Proposition 2).

Hence, the optimal combination of initial margin and default fund consists of the maximum possible level of initial margin along the incentive-compatible curve, i.e., $$I^{SB} = D - A_{CCP}(\mu_s - \mu_r)\left(\frac{q_r}{q_s}\right)p_c$$, and $$F^{SB} = 0$.$$

**Proof of Proposition 4**

Under Assumption 3, the first-order derivative of $$W(I; \hat{F}(I))$$ with respect to $$I$$ is given by

$$\frac{\partial}{\partial I} W(I; \hat{F}(I)) = -\beta \frac{\partial}{\partial I} \left[ I + \hat{F}(I) \right] - p_c \alpha \frac{\partial}{\partial I} \mathbb{E}^s \left[ \left( \frac{N_d(D - I)}{N} - \hat{F}(I) \right)^+ \right] = (\beta - p_c \alpha) \frac{1 - v_l^{(N)}(q_s)}{u_l^{(N)}(q_s)}.$$
We discuss the three possible cases below.

**Case 1:** When $\beta > p_c \alpha$,
\[ \frac{\partial}{\partial I} W(I; \hat{F}(I)) > 0. \]

Hence, $I^{SB} = D - \frac{A_{CCP}(\mu_e - \mu_s)}{(q_e - q_s)p_e}$, and $F^{SB} = 0$.

**Case 2:** When $\beta = p_c \alpha$,
\[ \frac{\partial}{\partial I} W(I; \hat{F}(I)) = 0. \]

Hence, any choice of initial margin is optimal as long as $I^{SB} \in \left[ 0, D - \frac{A_{CCP}(\mu_e - \mu_s)}{(q_e - q_s)p_e} \right]$.

**Case 3:** When $p_c \alpha > \beta$,
\[ \frac{\partial}{\partial I} W(I; \hat{F}(I)) < 0. \]

Hence, it is optimal to set $I^{SB} = 0$.

Taken together, the three cases above yield $I^{SB}$ in Eq. (20). Plugging $I^{SB}$ into Eq. (A7) gives Eq. (21). The equality $\hat{F} \left( I^{SB} = D - \frac{A_{CCP}(\mu_e - \mu_s)}{(q_e - q_s)p_e} \right) = 0$ follows directly from Proposition 2.

**Proof of Proposition 5**

By Eq.(15), the limit of $\psi(N - 1; I; F)$ directly yields the limit of $\hat{F}(I)$. Hence, we first compute the limit of $\psi(N - 1; I; F)$ as $N \to \infty$. From the definitions of $v_i^{(N)}$ and $u_i^{(N)}$ in Eq. (A5) and Eq. (A6),

\[ v_i^{(N)}(q_s) = \mathbb{P}(X \leq l) = \mathbb{P} \left( \sqrt{N-1} \left( \frac{X}{N-1} - q_s \right) \leq \sqrt{N-1} \left( \frac{l}{N-1} - q_s \right) \right), \]
\[ u_i^{(N)}(q_s) = \frac{\mathbb{P}(Y \leq l)}{1 - q_s} = \mathbb{P} \left( \sqrt{N} \left( \frac{Y}{N} - q_s \right) \leq \sqrt{N} \left( \frac{l}{N} - q_s \right) \right), \]

where $X$ follows a Binomial distribution with parameter $(N-1, q_s)$ and $Y$ follows a Binomial distribution with parameter $(N, q_s)$. By the central limit theorem, both $\sqrt{N-1} \left( \frac{X}{N-1} - q_s \right)$ and $\sqrt{N} \left( \frac{Y}{N} - q_s \right)$ converge in distribution to a Gaussian distribution with mean 0 and variance $q_s(1 - q_s)$. On the other hand, using the inequality $\frac{N}{D - 1} - 1 < l = \lfloor \frac{NF}{D - 1} \rfloor \leq \frac{NF}{D - 1}$ we can conclude that

\[ \lim_{N \to \infty} \sqrt{N-1} \left( \frac{l}{N-1} - q_s \right) = \lim_{N \to \infty} \sqrt{N} \left( \frac{l}{N} - q_s \right) = \infty \mathbb{1}_{\frac{N}{D-1} > q_s} + 0 \cdot \mathbb{1}_{\frac{N}{D-1} = q_s} + (-\infty) \cdot \mathbb{1}_{\frac{N}{D-1} < q_s}. \]

Using the fact that a zero-mean Gaussian distribution is smaller than $\infty$ with probability 1, and smaller than 0 with probability 1/2, we obtain that

\[ \lim_{N \to \infty} v_i^{(N)}(q_s) = \mathbb{1}_{\frac{N}{D-1} > q_s} + \frac{1}{2} \mathbb{1}_{\frac{N}{D-1} = q_s}, \quad \lim_{N \to \infty} u_i^{(N)}(q_s) = \frac{\mathbb{1}_{\frac{N}{D-1} > q_s} + \frac{1}{2} \mathbb{1}_{\frac{N}{D-1} = q_s}}{1 - q_s}. \]
Combining the above limiting results with (A4), we obtain that
\[
\lim_{N \to \infty} \psi(N - 1; I; F) = \left( \frac{-q_s(D - I) + F}{1 - q_s} \right) 1_{\frac{F}{q_s - I} > q_s} + \frac{1}{2} \left( \frac{-q_s(D - I) + F}{1 - q_s} \right) 1_{\frac{F}{q_s - I} = q_s},
\]
where we observe that \( \left( \frac{-q_s(D - I) + F}{1 - q_s} \right) 1_{\frac{F}{q_s - I} = q_s} = 0 \), hence \( \frac{F}{q_s - I} = q_s \) is a degenerate case.

Using Eq. (15) and (A14), we conclude that the limit of \( \hat{F}(I) \) converges to the solution of the equation:
\[
\left( \frac{-q_s(D - I) + F}{1 - q_s} \right) 1_{\frac{F}{q_s - I} > q_s} = D - I - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}.
\]
Hence, we obtain that
\[
\lim_{N \to \infty} \hat{F}(I) = D - I - \frac{(1 - q_s)A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c}.
\]
Finally, by Eq. (7) and Eq. (22), we obtain that as \( N \to \infty \), Eq. (23) holds. Since \( \frac{F}{D - I} > q_s \), we obtain that \( \lim_{N \to \infty} x^{SB}(I; N) > q_s \).

To obtain comparative statics on the limit of the “Cover x%” rule, we take derivatives:

- with respect to \( \gamma \):
  \[
  \frac{\partial x}{\partial \gamma} = -\frac{(\mu_s - \mu_r)(1 - q_s)(1 - p_c)D^2}{(q_r - q_s)(D - I)} < 0;
  \]

- with respect to \( \mu_s - \mu_r \) (while fixing \( q_r, q_s \)):
  \[
  \frac{\partial x}{\partial (\mu_s - \mu_r)} = \frac{(1 - q_s)D(1 + \gamma D(1 - p_c))}{(q_r - q_s)(D - I)} < 0;
  \]

- with respect to \( p_c \):
  \[
  \frac{\partial x}{\partial p_c} = \gamma D^2 \frac{(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)(D - I)} > 0.
  \]

This proves Proposition 5.

**Proof of Proposition 6**

With a convex CCP recapitalization cost, the objective function, \( W(I; \hat{F}(I)) \), is piecewise concave in \( I \) (rather than piecewise linear as in the baseline model). Specifically, from Lemma IA.4, for each \( l = 0, 1, ..., N - 1 \), the function \( W(I; \hat{F}(I)) \) is quadratic concave with continuous first-order derivative which is linearly decreasing in \( I \); hence, there exists a unique local optimum given by \( \min (\max (I_{l+1}, I^*_l), I_l) \), where \( I^*_l \) is the unique solution of...
\[ \frac{\partial}{\partial I} W(I; \hat{F}(I)) = 0, \]  
for each \( l \). From Eq. (A30), the explicit expression of \( I^*_l \) is given by

\[
I^*_l = D - \frac{2p_c \alpha N \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c a_t^N(q_s)} - \beta}{2p_c \alpha N \mathbb{E}^s \left( \frac{N - N_d}{N} + \frac{1}{u_t^N(q_s)} \right)^2} \left( 1 - v_t^N(q_s) \right). \tag{A15}
\]

Finally, the function \( W(I; \hat{F}(I)) \) is continuous and convex at the kinks \( \{ I_l \} \), which suggests that the global optimum cannot be at a kink, unless the kink corresponds to the maximum or minimum possible \( I \). Together, we conclude that the global optimum can occur either at an interior local optimum in each interval \( (I_{l+1}, I_l) \), or at the upper bound \( I = D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c} \) or lower bound \( I = 0 \). This completes the proof of Eq. (24).

**Proof of Proposition 7**

Denote by \( \tilde{N}_d = \sum_{i=1}^{N} \mathbf{1}_i \) defaults the realized number of defaulted members (excluding the big member) when the credit event occurs. To compute the expected refund of each member, let us denote by \( q_a \) the failure probability of a big member who makes risk-management choice \( a \). We refer to all the other members as “small members” and denote by \( q_s \) the failure probability of a small member who makes risk-management choice \( a \).

We start by analyzing the risk-management decision of a small member. With probability \( q_a \), the big member fails, and the refund to a small member is \( \left( F - \frac{(\tilde{N}_a + K)(D - I - F)}{N - 1 - \tilde{N}_d} \right)^+ \).

With probability \( 1 - q_a \), the big member succeeds, and the refund to a small member is \( \left( F - \frac{\tilde{N}_a(D - I - F)}{N - 1 - \tilde{N}_d} \right)^+ \). Combining these two situations, any small member \( i \geq 2 \) chooses to hedge if (the right-hand side is the necessary refundable collateral analogous to Eq. (15)):

\[
I + q_a^\text{big} \mathbb{E}^s \left[ \left( F - \frac{(\tilde{N}_a + K)(D - I - F)}{N - 1 - \tilde{N}_d} \right)^+ \right] + (1 - q_a^\text{big}) \mathbb{E}^s \left[ \left( F - \frac{\tilde{N}_a(D - I - F)}{N - 1 - \tilde{N}_d} \right)^+ \right] \geq D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c}. \tag{A16}
\]

We rewrite (A16) by introducing the functions \( \Psi^1(I; F) \) and \( \Psi^2(I; F) \), and define the cutoff value \( \hat{F}_{\text{small}}(I) \) as satisfying

\[
I + q_s \Psi^1(I; \hat{F}_{\text{small}}(I)) + (1 - q_s) \Psi^2(I; \hat{F}_{\text{small}}(I)) = D - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s) p_c}. \tag{A17}
\]

Using an analogous reasoning as in Lemma IA.1, we conclude that the left-hand side of (A17) is piecewise linear and strictly increasing in \( F \). Also, the left-hand side equals \( D \) when \( F = D - I \). Hence, we can conclude that for a given \( I \), \( \hat{F}_{\text{small}}(I) \) is uniquely determined by Eq. (A17) and that \( \hat{F}_{\text{small}}(I) < D - I \).

Using the dominated convergence theorem, we can take the derivative of the left-hand
side in Eq. (A16), and obtain

\[
\begin{align*}
\frac{\partial \Psi^1}{\partial F} - \frac{\partial \Psi^1}{\partial I} &= \mathbb{P}^s \left( F > \frac{(\tilde{N}_d + K)(D - I - F)}{N - 1 - \tilde{N}_d} > 0 \right), \\
\frac{\partial \Psi^2}{\partial F} - \frac{\partial \Psi^2}{\partial I} &= \mathbb{P}^s \left( F > \frac{\tilde{N}_d(D - I - F)}{N - 1 - \tilde{N}_d + K} \right) \in (0, 1).
\end{align*}
\] (A18)  

(A19)

Since \( K > 1 \), it always holds

\[
1 > \mathbb{P}^s \left( F > \frac{\tilde{N}_d(D - I - F)}{N - 1 - \tilde{N}_d + K} \right) > \mathbb{P}^s \left( F > \frac{(\tilde{N}_d + K)(D - I - F)}{N - 1 - \tilde{N}_d} \right) > 0. \] (A20)

Applying the implicit function theorem to Eq. (A17), we obtain

\[
\left( q_s \frac{\partial \Psi^1 \left( I; \hat{F}_{\text{small}}(I) \right)}{\partial F} + (1 - q_s) \frac{\partial \Psi^2 \left( I; \hat{F}_{\text{small}}(I) \right)}{\partial F} \right) dF + \\
\left( 1 + q_s \frac{\partial \Psi^1 \left( I; \hat{F}_{\text{small}}(I) \right)}{\partial I} + (1 - q_s) \frac{\partial \Psi^2 \left( I; \hat{F}_{\text{small}}(I) \right)}{\partial I} \right) dI = 0,
\]

which yields

\[
\frac{d\hat{F}_{\text{small}}(I)}{dI} = -q_s \left( 1 + \frac{\partial \Psi^1 \left( I; \hat{F}_{\text{small}}(I) \right)}{\partial I} \right) + (1 - q_s) \left( 1 + \frac{\partial \Psi^2 \left( I; \hat{F}_{\text{small}}(I) \right)}{\partial I} \right) < -1, \quad (A21)
\]

where the last inequality follows from (A18)–(A19).

Next, we analyze the risk-management decision of the big member. Given the expected refund, the big member \( i = 1 \) chooses to hedge if

\[
I + \mathbb{E}^s \left[ \left( F - \frac{\tilde{N}_d(D - I - F)}{N - 1 - \tilde{N}_d + K} \right)^+ \right] \geq D - \frac{ACCP}{(q_r - q_s)p_c} \left( \mu_s - \mu_r \right). \] (A22)

Using the definition of the functions \( \Psi^1 (I; F) \) and \( \Psi^2 (I; F) \) in (A16)-(A17), we introduce the cutoff value \( \hat{F}_{\text{big}}(I) \) as satisfying

\[
I + \Psi^2 (I; \hat{F}_{\text{big}}(I)) = D - \frac{ACCP}{(q_r - q_s)p_c} \left( \mu_s - \mu_r \right). \] (A23)

Similarly to the analysis of \( \hat{F}_{\text{small}} \), we know that for a given \( I \), \( \hat{F}_{\text{big}}(I) \) is uniquely determined by Eq. (A23) and that \( \hat{F}_{\text{big}}(I) < D - I \). Moreover, for any given \( I \) and \( F \), \( \Psi^1 < \Psi^2 \) and both \( \Psi^1 \) and \( \Psi^2 \) are strictly increasing in \( F \). Hence, we can conclude that \( 0 \leq \hat{F}_{\text{big}}(I) \leq \hat{F}_{\text{small}}(I) < D - I \). Combining (A16)–(A17) and (A22)–(A23), we deduce Eq. (25).
We apply the implicit function theorem to Eq. (A23) and use (A19) to obtain
\[
\frac{d\hat{F}_{\text{big}}(I)}{dI} = -1 + \frac{\partial \Psi^2(I; \hat{F}_{\text{big}}(I))}{\partial I} + 1 + \frac{\partial \Psi^2(I; \hat{F}_{\text{big}}(I))}{\partial I} < -1.
\] (A24)

Since both \(\frac{\partial \Psi^1}{\partial I}\) and \(\frac{\partial \Psi^2}{\partial I}\) are step functions of \(I\), similar to Proposition 2, we conclude that both \(\hat{F}_{\text{small}}(I)\) and \(\hat{F}_{\text{big}}(I)\) are piecewise linear and strictly decreasing with \(I\). Further, for given \(I\) and \(F\), the denominator of the right-hand side in (A21) is smaller than that in (A24) because of the inequality (A20). This leads to the conclusion that
\[
\frac{d\hat{F}_{\text{small}}(I)}{dI} < \frac{d\hat{F}_{\text{big}}(I)}{dI} < -1.
\]

**Proof of Proposition 8**

As in Proposition 4, the proof follows directly from Proposition 7 by discussing the three different cases arising from comparing \(\beta\) with \(p_c\alpha\).

**Proof of Proposition 9**

In the limiting case of a large clearing network, as \(N \to \infty\) and \(K/N \to 0\), we have
\[
\lim_{N \to \infty} \frac{\hat{N}_d}{N-1} = \lim_{N \to \infty} \frac{\hat{N}_d + K}{N-1} = \lim_{N \to \infty} \frac{\hat{N}_d - K}{N-1} = q_s.
\]

It follows from convergence in probability that
\[
\lim_{N \to \infty} \mathbb{E}^s \left[ \left( F - \frac{(\hat{N}_d + K)(D - I - F)}{N - 1 - \hat{N}_d} \right)^+ \right] = \lim_{N \to \infty} \mathbb{E}^s \left[ \left( F - \frac{\hat{N}_d(D - I - F)}{N - 1 - \hat{N}_d + K} \right)^+ \right] = \lim_{N \to \infty} \psi(I; F).
\]

Using the definition of the functions \(\Psi^1(I; F)\) and \(\Psi^2(I; F)\), we obtain
\[
\lim_{N \to \infty} \Psi^1(I; F) = \lim_{N \to \infty} \Psi^2(I; F) = \lim_{N \to \infty} \psi(I; F),
\]
where \(\lim_{N \to \infty} \psi(I; F)\) is given by Eq. (A14). Since the expected refund of the default fund converges to the same expression as in the baseline model, we must have that
\[
\lim_{N \to \infty} \hat{F}_{\text{small}}(I) = \lim_{N \to \infty} \hat{F}_{\text{big}}(I) = \lim_{N \to \infty} \hat{F}(I).
\]

Since the threshold values of required default fund, both for the big and small members, coincide with those in Proposition 5, the optimal cover ratio has the same expression, given by Eq. (23). This proves Proposition 9.

Furthermore, suppose the clearinghouse consists of several big members with heterogeneous sizes. We can argue that the above result still holds as long as the total mass of big members is zero in the limiting case of many members. This result can be obtained by sequentially adding more big members to the system: In each step, we already know that the members will behave as if the system were size-homogeneous.
Proof of Proposition 10

We have shown in Proposition 5 that, in the limiting case of a large number of members,

$$\lim_{N \to \infty} F(I^{SB}) = D - I^{SB} - \frac{(1 - q_s) A_{\text{CCP}}(\mu_s - \mu_r)}{(q_r - q_s)p_c}.$$ 

Plugging the expression above into the expected loss of the CCP, we have

$$\lim_{N \to \infty} E_s^s \left[ \left( \frac{N_d}{N}(D - I^{SB}) - \hat{F}(I^{SB}) \right)^+ \right] = \lim_{N \to \infty} E_s^s \left[ \left( \frac{(1 - q_s) A_{\text{CCP}}(\mu_s - \mu_r)}{(q_r - q_s)p_c} - \frac{N - N_d}{N}(D - I^{SB}) \right)^+ \right] = (1 - q_s) \left( \frac{A_{\text{CCP}}(\mu_s - \mu_r)}{(q_r - q_s)p_c} - (D - I^{SB}) \right)^+.$$ 

Because of Eq. (15), we must have that $D - I^{SB} \geq \frac{A_{\text{CCP}}(\mu_s - \mu_r)}{(q_r - q_s)p_c}$; we conclude that

$$\lim_{N \to \infty} E_s^s \left[ \left( \frac{N_d}{N}(D - I^{SB}) - \hat{F}(I^{SB}) \right)^+ \right] = 0.$$ 

Intuitively, Proposition 5 shows that the optimal “Cover $x\%$” rule when the number of members is large converges to a positive fraction greater than $q_s$, i.e., $\lim_{N \to \infty} x^{SB}(I; N) = \frac{\hat{F}(I^{SB}; N)}{D - I^{SB}} > q_s$. Hence, when infinite members are pooling risks together, the law of large numbers holds; their idiosyncratic default risk is fully diversified from the CCP’s perspective, and the total default funds are always sufficient to cover the total shortfalls, suggesting that the expected recapitalization cost of the CCP is zero.
Internet Appendix

In this Internet Appendix, we present auxiliary lemmas that are used to establish the technical results in the main paper.

**Lemma IA.1** The term \( \psi(g; I; F) \) given in Eq. (13) admits the following explicit expression:

\[
\psi(g; I; F) = \sum_{k=N-1-[\frac{NF}{D-I}]}^{N-1} f_g(k) \left( F - \frac{N-(k+1)}{k+1}(D-I-F) \right), \tag{A25}
\]

where \( g = 0, 1, \ldots, N - 1 \), \([ \cdot ]\) denotes the floor function, and for each \( k \),

\[
f_g(k) = \sum_{m=0}^{k} \binom{g}{m} (1 - q_s)^m q_s^{g-m} \binom{N-1-g}{k-m} (1 - q_r)^{k-m} q_r^{N-1-g-(k-m)} \tag{A26}
\]

is a positive constant. For \( I + F < D \), the function \( \psi(g; I; F) \) is strictly increasing in \( g \), is piecewise linear and strictly increasing in both \( I \) and \( F \). Moreover, \( 0 < \psi(g; I; F) < F \) and \( \psi(g; I; D-I) = D-I \).

**Proof.** To see how Eq. (A25)–(A26) are obtained, notice first that the term \( F - \frac{N-(k+1)}{k+1}(D-I-F) \) represents the amount of refund to a surviving member \( i \) in the event that \( k \) among the remaining \((N - 1)\) members survive. The term \( f_g(k) \) is the probability of such an event when \( g \) of the remaining \((N - 1)\) members choose to hedge and \((N - 1 - g)\) of them choose not to hedge. If the number of survivors among the remaining \((N - 1)\) members is too small, i.e., \( k < N - 1 - \left[ \frac{NF}{D-I} \right] \), the total default funds are not sufficient to cover the total default shortfalls, and thus the refund to member \( i \) is zero.

The function \( \psi(g; I; F) \) depends on \( g \) through the term \( f_g(k) \), the probability that \( k \) among the remaining \((N - 1)\) members survive when \( g \) of the \((N - 1)\) members choose to hedge. Since \( q_s < q_r \), \( f_g(k) \) is strictly increasing in \( g \), and so \( \psi(g; I; F) \) is strictly increasing in \( g \).

From Eq. (A25), \( \psi(g; I; F) \) is linear and strictly increasing in \( I \) and \( F \) within any subinterval \( F \in \left( \frac{(N-k)(D-I)}{N}, \frac{(N-k)(D-I)}{N} \right) \) with \( k = 0, 1, \ldots, N - 1 \). Hence, we conclude that \( \psi(g; I; F) \) is piecewise linear in \( I \) and \( F \). The nonnegative random variable \( F - \frac{N-(k+1)}{k+1}(D-I-F) \) is almost surely continuous in \( I \) and \( F \) and bounded by \( (D-I) \).

By the dominated convergence theorem, the expected value of this random variable, i.e., \( \psi(g; I; F) \), is continuous. Therefore, the function \( \psi(g; I; F) \) is piecewise linear and strictly increasing in \( I \) and \( F \).

Finally, recall from Eq. (A25), \( \psi(g; I; F) \) measures the expected refund of the segregated default fund, and \( F - \psi(g; I; F) \) is the expected contribution of a surviving member \( i \) towards other members’ default losses. With a positive probability, at least one member defaults, and the expected contribution of a surviving member \( i \) towards other members’ default losses is positive. Similarly, with a positive probability, no members default, and member \( i \) receives a full refund from the CCP. This suggests that the expected refund must also be positive. Hence, for any fixed \( g \), we have \( 0 < \psi(g; I; F) < F \). It also follows from Eq. (A25) that \( \psi(g; I; D-I) = \sum_{k=0}^{N-1} f_g(k)(D-I) = D-I \).
Lemma IA.2  The function \( h(F) := -(\beta + p_c) F + (1 - q_s) p_c \psi(N - 1; F) \) is strictly decreasing in \( F \) over the interval \([0, D]\).

**Proof.** From the definition of \( \psi(g; I; F) \) given in Eq. (13),

\[
\psi(N - 1; F) = E^s \left[ \left( F - \frac{N_d (D - F)}{N - N_d} \right)^+ \right] = E^s \left[ \left( \frac{N (F - D)}{N - N_d} + D \right)^+ \right].
\]

Since \( \left( \frac{N (F - D)}{N - N_d} + D \right)^+ \) is convex in \( F \), we can conclude that \( \psi(N - 1; F) \) is convex as well. It follows that \( h(F) \) is convex in the interval \([0, D]\).

To complete the proof, it suffices to show that \( h'(D-) = 0 \). To this end, we focus on the sub-interval \( F \in \left[ \frac{N - 1}{N}, D \right] \). It follows that

\[
\psi(N - 1; F) = D + (F - D) E^s \left[ \frac{N}{N - N_d} \right] (N - N_d) \geq 1.
\]

Since \( (N - N_d - 1) \) follows a binomial distribution with parameter \( (N - 1, 1 - q_s) \), we have

\[
\psi(N - 1; F) = D + (F - D) \sum_{k=0}^{N-1} \binom{N}{1+k} (1-q_s)^k q_s^{N-1-k} = D + (F - D) \sum_{m=1}^{N} \binom{N}{m} (1-q_s)^m q_s^{N-m} = D + (F - D) \frac{1 - q_s^N}{1 - q_s}.
\]

Taking derivatives with respect to \( F \), we obtain that

\[
h'(D-) = -(\beta + p_c) + (1 - q_s) p_c \frac{1 - q_s^N}{1 - q_s} = -(\beta + p_c q_s^N) < 0.
\]

Therefore \( h(F) \) is strictly decreasing in \( F \) over \([0, D]\). \(\blacksquare\)

Lemma IA.3 Define the kinks as \( \{ l = \frac{N \hat{F}(I)}{D - I} = 0, 1, ..., N - 1 \} \). The function \( \hat{F}(I) \) is piecewise linear in \( I \) with negative slopes, and is continuous and concave at the kinks. The function \( E^s \left[ \frac{N_d (D - I)}{N} - \hat{F}(I) \right]^+ \) is piecewise linear in \( I \) with positive slopes and is continuous and convex at the kinks. In addition, the following relation holds:

\[
\frac{\partial}{\partial I} \left( I + \hat{F}(I) \right) = - \frac{\partial}{\partial I} E^s \left[ \frac{N_d (D - I)}{N} - \hat{F}(I) \right]^+ = - \frac{1 - v_i^{(N)}(q_s)}{u_i^{(N)}(q_s)}.
\]

**Proof.** From the expression of \( \hat{F}(I) \) in Eq. (A7),

\[
\frac{\partial}{\partial I} \left( I + \hat{F}(I) \right) = - \frac{1 - v_i^{(N)}(q_s)}{u_i^{(N)}(q_s)} < 0.
\]
Hence, the function $\hat{F}(I)$ is piecewise linear with negative slopes (less than $-1$). Moreover, for each $l = 0, 1, ..., N - 1$, the kink $I_l$ is uniquely defined by $l = \frac{N\hat{F}(I_l)}{D-I_l}$ because $\hat{F}(I_l)$ is a linear function of $I_l$.

Next, we analyze the behavior of the function $\hat{F}(I)$ around the kinks. We begin by showing that it is continuous at kink $I_l$. From the expression of $\hat{F}(I)$ in Eq. (A7),

$$\hat{F}(I_l^+)-\hat{F}(I_l^-) = \left(1 - \frac{v_l(N)}{u_{l-1}(q_s)} - \frac{v_l(N)}{u_{l}(q_s)}\right) - \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \left(\frac{1}{u_{l-1}(q_s)} - \frac{1}{u_{l}(q_s)}\right).$$

From the definitions of $v_l(N)$ and $u_l(N)$ in Eq. (A5)–(A6), we obtain

$$v_l(N) - v_{l-1}(N) = \left(\frac{N-1}{l}\right)q_s^l(1-q_s)^{N-1-l} = \left(1 - \frac{l}{N}\right)\left(\frac{u_l(N) - u_{l-1}(N)}{u_l(N)}\right).$$

Using this result and after some algebra, we obtain

$$\left(1 - \frac{v_{l-1}(N)}{u_{l-1}(q_s)} - \frac{v_l(N)}{u_l(N)}\right) = \left(1 - \frac{1}{\frac{l}{N}} + \frac{1-v_l(N)}{u_l(N)}\right) \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \frac{1}{u_l(N)}.$$  \hspace{1cm} (A28)

Furthermore, from the definition of the kink $l = \frac{N\hat{F}(I)}{D-I}$ and Eq. (A7), we have

$$\left(1 - \frac{l}{N} + \frac{1-v_l(N)}{u_l(N)}\right) = \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \frac{1}{u_l(N)}.$$  \hspace{1cm} (A28)

Plugging the above expression into Eq. (A28), we have

$$\left(1 - \frac{v_{l-1}(N)}{u_{l-1}(q_s)} - \frac{v_l(N)}{u_l(N)}\right) \frac{A_{CCP}(\mu_s - \mu_r)}{(q_r - q_s)p_c} \left(\frac{1}{u_{l-1}(q_s)} - \frac{1}{u_l(N)}\right),$$

which completes the proof that $\hat{F}(I_l^+)-\hat{F}(I_l^-) = 0$, i.e., the function $\hat{F}(I)$ is continuous.

We then take the partial derivatives of the function $\hat{F}(I)$ around the kinks: \hspace{1cm} \hspace{1cm}

$$\frac{\partial}{\partial I} \hat{F}(I_{l-}) = -1 - \frac{v_l(N)}{u_l(N)} = -1 - \frac{1-q_s}{\frac{\mathbb{P}^*(X > l|X \sim \text{Binom}(N-1, q_s))}{\mathbb{P}^*(Y \leq l|Y \sim \text{Binom}(N, q_s))}};$$

$$\frac{\partial}{\partial I} \hat{F}(I_{l+}) = -1 - \frac{v_{l-1}(N)}{u_{l-1}(q_s)} = -1 - \frac{1-q_s}{\frac{\mathbb{P}^*(X > l-1|X \sim \text{Binom}(N-1, q_s))}{\mathbb{P}^*(Y \leq l-1|Y \sim \text{Binom}(N, q_s))}}.$$
Comparing the derivatives around the kink, we have

\[
\frac{\mathbb{P}^s(X > l - 1 | X \sim \text{Binom}(N - 1, q_s))}{\mathbb{P}^s(Y \leq l - 1 | Y \sim \text{Binom}(N, q_s))} > \frac{\mathbb{P}^s(X > l | X \sim \text{Binom}(N - 1, q_s))}{\mathbb{P}^s(Y \leq l | Y \sim \text{Binom}(N, q_s))} \Rightarrow \frac{\partial}{\partial I} \hat{F}(I^-) > \frac{\partial}{\partial I} \hat{F}(I^+) .
\]

Hence, we have shown that function \( \hat{F}(I) \) is piecewise linear in \( I \) with negative slopes, and is continuous and concave at the kinks.

Finally, we analyze the behavior of the function \( \mathbb{E}^s \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] \). Given binomial random variables \( X \sim \text{Binom}(N - 1, q_s) \) and \( Y \sim \text{Binom}(N, q_s) \), we have

\[
q_s \mathbb{P}^s(X \leq l - 1) + (1 - q_s) \mathbb{P}^s(X \leq l) = \mathbb{P}^s(Y \leq l).
\]

Plugging the defining expressions of \( v_{i-1}^{(N)}(q_s) \), \( v_i^{(N)}(q_s) \), and \( u_i^{(N)}(q_s) \) in the expression above yields

\[
q_s v_{i-1}^{(N)}(q_s) + (1 - q_s) v_i^{(N)}(q_s) = (1 - q_s) u_i^{(N)}(q_s).
\]

Using the above equality, we calculate the left derivative of \( \mathbb{E}^s \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] \) with respect to \( I \) as follows:\(^{35}\)

\[
- \frac{\partial}{\partial I} \mathbb{E}^s \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] = \sum_{k=l+1}^{N} \binom{N}{k} q^k_s (1 - q_s)^{N-k} \left( \frac{k}{N} + \frac{\partial}{\partial I} \hat{F}(I^-) \right)
\]

\[
=q_s \left( 1 - v_{i-1}^{(N)}(q_s) \right) - \left( 1 + \frac{1 - v_i^{(N)}(q_s)}{u_i^{(N)}(q_s)} \right) \left( 1 - (1 - q_s) u_i^{(N)}(q_s) \right)
\]

\[
=q_s - q_s v_{i-1}^{(N)}(q_s) - 1 - \frac{v_i^{(N)}(q_s)}{u_i^{(N)}(q_s)} + (1 - q_s) \left( 1 + u_i^{(N)}(q_s) - v_i^{(N)}(q_s) \right)
\]

\[
= - \frac{1 - v_i^{(N)}(q_s)}{u_i^{(N)}(q_s)} = \frac{\partial}{\partial I} \left( I + \hat{F}(I) \right).
\]

This proves Eq. (A27). Hence, we conclude that the function \( \mathbb{E}^s \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] \) is piecewise linear in \( I \) with positive slopes and is convex at the kinks. The operator \( \mathbb{E}^s [(\cdot)^+] \) preserves functional continuity, so function \( \mathbb{E}^s \left[ \left( \frac{N_d(D-I)}{N} - \hat{F}(I) \right)^+ \right] \) is also continuous. \( \blacksquare \)

\(^{35}\)The second equality uses \( \sum_{k=l+1}^{N} \binom{N}{k} q^k_s (1 - q_s)^{N-k} k = Nq_s \left( 1 - v_{i-1}^{(N)}(q_s) \right) \) and \( \sum_{k=l+1}^{N} \binom{N}{k} q^k_s (1 - q_s)^{N-k} = 1 - (1 - q_s) u_i^{(N)}(q_s) \).
Lemma IA.4 Given the function $\hat{F}(I)$ in Eq. (A7), define the function $W(I; \hat{F}(I))$ as

$$W(I; \hat{F}(I)) := A_{CCP}p_s - \beta(I + \hat{F}(I)) - \frac{p_s\alpha}{N}E^s \left[ (N_d(D - I) - N\hat{F}(I))^+ \right]^2 - p_cD. \quad (A29)$$

Define the kinks as $I_i : I_i > I_{i+1}, l = \frac{N\hat{F}(I_l)}{D - I_l} = 0, 1, ..., N - 1$. For each $l = 0, 1, ..., N - 1$, the function $W(I; \hat{F}(I))$ is quadratic concave with continuous first-order derivative linearly decreasing in $I$. At the kinks, the function $W(I; \hat{F}(I))$ is continuous and convex.

**Proof.** From Lemma IA.3, for $l = 0, 1, ..., N - 1$, kink $I_i$ is uniquely defined as $\hat{F}(I_l)$ is a piecewise linear function of $I_i$. Also, $-\beta(I + \hat{F}(I))$ is piecewise linear with positive slopes, and is continuous and convex at the kinks. Next, we show that $-\frac{p_s\alpha}{N}E^s \left[ (N_d(D - I) - N\hat{F}(I))^+ \right]^2$ is a decreasing and concave function except at the kinks.

$$\frac{\partial}{\partial I} -\frac{p_s\alpha}{N}E^s \left[ (N_d(D - I) - N\hat{F}(I))^+ \right]^2 = -2p_c\alpha E^s \left[ (N_d(D - I) - N\hat{F}(I))^+ \left( -\frac{N_d}{N} - \frac{\partial \hat{F}(I)}{\partial I} \right) \right] < 0,$$

where the last inequality follows from the following inequality: $-\frac{N_d}{N} - \frac{\partial \hat{F}(I)}{\partial I} = \frac{N - N_d}{N} + \frac{1 - v_i^{(N)}(q_*)}{u_i^{(N)}(q_*)} > 0$. Moreover,

$$\frac{\partial^2}{\partial I^2} -\frac{p_s\alpha}{N}E^s \left[ (N_d(D - I) - N\hat{F}(I))^+ \right]^2 = -2p_c\alpha N E^s \left( -\frac{N_d}{N} - \frac{\partial \hat{F}(I)}{\partial I} \right)^2 < 0.$$

We can further obtain an explicit expression for $\frac{\partial}{\partial I} W(I; \hat{F}(I))$ using Lemma IA.3.

$$\frac{\partial}{\partial I} W(I; \hat{F}(I)) = -\beta \left( 1 + \frac{\partial \hat{F}(I)}{\partial I} \right) - 2p_c\alpha E^s \left[ (N_d(D - I) - N\hat{F}(I))^+ \left( -\frac{N_d}{N} - \frac{\partial \hat{F}(I)}{\partial I} \right) \right]$$

$$= \beta \frac{1 - v_i^{(N)}(q_*)}{u_i^{(N)}(q_*)} - 2p_c\alpha E^s \left[ N_d \left( -\frac{N_d}{N} - \frac{\partial \hat{F}(I)}{\partial I} \right) \right] _{N_d \geq 1 + l} (D - I) + 2p_c \alpha \hat{F}(I) N \frac{1 - v_i^{(N)}(q_*)}{u_i^{(N)}(q_*)}$$

$$= \left( \beta - 2p_c\alpha N \frac{ACC\mu - \mu_r}{p_c u_i^{(N)}(q_*)} \right) \frac{1 - v_i^{(N)}(q_*)}{u_i^{(N)}(q_*)} + 2p_c\alpha N E^s \left( -\frac{N_d}{N} - \frac{\partial \hat{F}(I)}{\partial I} \right)^2 (D - I). \quad (A30)$$

We can conclude that, for each $l$, the function $\frac{\partial}{\partial I} W(I; \hat{F}(I))$ is a continuous, linear function of $I$ with negative slopes. Therefore, for each $l$, there exists a unique local optimum given by $\min(\max(I_{l+1}, I^*_l), I_l)$, where $I^*_l$ is the unique solution of $\frac{\partial}{\partial I} W(I; \hat{F}(I)) = 0$.

To analyze the behavior of the function at the kinks, we invoke the dominated convergence
theorem:
\[
\frac{\partial}{\partial I} W(I_t^+; \hat{F}(I_t^+)) = -\beta \left( 1 + \frac{\partial \hat{F}(I_t^+)}{\partial I} \right) - 2p_c \alpha E_s \left[ \left( N_d(D - I_t) - N\hat{F}(I_t) \right)^+ \left( \frac{-N_d}{N} - \frac{\partial \hat{F}(I_t^+)}{\partial I} \right) \right];
\]
\[
\frac{\partial}{\partial I} W(I_t^-; \hat{F}(I_t^-)) = -\beta \left( 1 + \frac{\partial \hat{F}(I_t^-)}{\partial I} \right) - 2p_c \alpha E_s \left[ \left( N_d(D - I_t) - N\hat{F}(I_t) \right)^+ \left( \frac{-N_d}{N} - \frac{\partial \hat{F}(I_t^-)}{\partial I} \right) \right].
\]

Taking the difference of the derivatives at the kinks, we obtain
\[
\frac{\partial}{\partial I} W(I_t^+; \hat{F}(I_t^+)) - \frac{\partial}{\partial I} W(I_t^-; \hat{F}(I_t^-)) = \left( -\beta + 2p_c \alpha E_s \left[ \left( N_d(D - I_t) - N\hat{F}(I_t) \right)^+ \right] \right) \left( \frac{\partial \hat{F}(I_t^+)}{\partial I} - \frac{\partial \hat{F}(I_t^-)}{\partial I} \right).
\]

From Lemma IA.3, \( \hat{F}(I) \) is continuous and concave at the kinks. Hence, \( W(I; \hat{F}(I)) \) is continuous and convex at the kinks. ■